Extensions of groups by weighted Steiner loops

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A loop is a quasigroup with identity element e.

A binary system $(L, \cdot)$ is called a loop if there exists an element $e' \in L$ such that $x = e' \cdot x = x \cdot e'$ holds for all $x \in L$ and for all given $a, b \in L$ the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution , which we denote by $y = a \backslash b$ and $x = b/a$. The left and right translations $\lambda_a : y \mapsto a \cdot y : L \to L$ and $\rho_a : y \mapsto y \cdot a : L \to L$, $a \in L$ are permutations of L.

Any group which is an extension G of a normal subgroup N by a group K is determined by two functional equations describing the action of K on N and ensuring the associative law in G by the choice of a system of representatives in G for the factor group G/N isomorphic to K.

See: O. Schreier, Über die Erweiterung von Gruppen I, Monatsh. Math. Phys. 34, 1926, 165-180.

O. Schreier, Über die Erweiterung von Gruppen II, Abhandl. Math. Sem. Hamburg. Univ. 6, 1926, 321-346.

In the paper P. Nagy and K. Strambach: Schreier loops, Czechoslovak Mathematical Journal 58, 2008, pp. 759-786, it was studied a variation of extensions which yields loops as extensions of groups by loops such that these extensions with interesting weak-associativity properties can be described by the same type of identities as in the group case.

Extension of a group A by a loop S

Let $(L, \cdot)$ be an extension of a group A by a loop $(S, *)$

(1) $$1 \longrightarrow A \longrightarrow L \longrightarrow S \longrightarrow e$$

which is defined on the set $S \times A$ by the multiplication

(2) $$(x, \xi) \cdot (y, \eta) = (x * y, f(x, y)\xi^{T(y)}\eta),$$

where $f : S \times S \to A$ is a map with $f(x, e) = f(e, y) = 1 \in A$ for all $x, y \in S$ and $T : S \to \mathrm{Aut}(A)$ is a function of S into the automorphism group of A with $T(e) = \mathrm{Id}$. The identity element of the loop L is $(e, 1)$. We call the loop L Schreier loop.

We give concrete description for all extensions (1) such that the loop S is a weighted Steiner loop and for all $y \in S$ one has $T(y) = Id$.

Motivations to consider such extensions:

1. Relations to code theory:

Special types of extensions L of the group of order 2 by finite Steiner loops, they are called code loops, are associated to doubly even binary codes.

R.L. Griess, Code loops, Journal of Algebra, 100, 1986, pp. 224-234,

T. Hsu, Explicit construction of code loops as centrally twisted products, Math. Proc. Cambridge Philos. Soc., 128, 2000, 223-232,

G. Nagy, Direct construction of code loops, Discrete Math., 308, 2008, pp. 5349-5357.

2. Motivation: K. Strambach, I. Stuhl: Oriented Steiner loops, Beitrắge zur Algebra and Geometrie, 54, 2013, 131-145.
They have been studied another type of extensions of the group of order 2 by Steiner loops, such that the corresponding Steiner triple system has an orientation.

Definition

A Steiner triple system $\mathfrak{S}$ is an incidence structure consisting of points and blocks such that every two distinct points are contained in precisely one block and any block has precisely three points. An oriented Steiner triple system $(\mathfrak{S}, T)$ is a Steiner triple system $\mathfrak{S}$ such that on the set T of blocks for each block a cyclic order is given.

With a Steiner triple system $\mathfrak{S}$ is associated a loop $(S(\mathfrak{S}), *)$ such that the elements of $S(\mathfrak{S}) \setminus \{e\}$ are the points of the Steiner triple system, the product $a * b$ is the third points of the block determined by a, b and $a * a = e$ for all $a \in \mathfrak{S}$.

This loop is a Steiner loop, i. e. it has the properties $x * y = y * x$ and $x * (x * y) = y$ for all $x, y \in S(\mathfrak{S})$. Putting $y = e$ into the last identity we have $x^2 = e$ for all $x \in S(\mathfrak{S})$. Hence a Steiner loop is a commutative loop of exponent 2 which has the inverse property and the bijections $\lambda_a = \rho_a$, $a \in S(\mathfrak{S})$, are involutions.

The orientation defines the factor system of the oriented Steiner loop such that if a_1, a_2 are distinct points determined the oriented block (a_1, a_2, a_3), then $f(a_1, a_2) = 1$ and $f(a_2, a_1) = -1$, moreover one has $f(x, x) = 1$, respectively -1 for all $x \in S \setminus \{e\}$. In particular they are characterized the oriented Steiner loop if the Steiner triple system is the point-line design of a projective geometry over $GF(2)$, as well as that of the affine plane of order 3.

Definition

A weighted Steiner loop (S, h) *is a Steiner loop* S *and a mapping* $h : S \setminus \{e\} \to A$*, where* A *is a group.*

We consider extensions L of groups A by weighted Steiner loops (S, h) such that for the factor system $f : S \times S \to A$ one has $f(x, y) = h(x)h(y)$ for all $x \neq y \in S \setminus \{e\}$. Such a loop L is called Steiner-like loop.

Á. Figula, K. Strambach, Extensions of groups by weighted Steiner loops, Results Math. 59, 2011, 251-278.

Definition

A loop $(L, \cdot)$ is power-associative if $(\langle a \rangle, \cdot)$ is associative for each $a \in L$.

Proposition

A Steiner-like loop L is power-associative if and only if the set $\{f(x,x);\ x \in S\}$ is contained in the centre of A.

In general, if S is a finite Steiner loop with n elements and if we take for the group A an extension of an abelian group Z by a finitely generated group D, then mapping $x \mapsto h(x)$, $x \in S \setminus \{e\}$, where $h(x)$ is a generator of D, we obtain a Steiner-like loop L, if the number of the generators of D does not exceed $n-1$ and the set $\{h(x); x \in S \setminus \{e\}\}$ contains a set of generators of D.

Definition

A loop L is right alternative, respectively left alternative if $(y \cdot x) \cdot x = y \cdot (x \cdot x)$, *respectively* $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ *holds for all* $x, y \in L$.

A loop L satisfies the left inverse, respectively the right inverse, respectively the cross inverse property if for all $x, y \in L$

$$x^{\lambda} \cdot (x \cdot y) = y, \ \text{respectively } (y \cdot x) \cdot x^{\rho} = y,$$

$$\text{respectively } (x \cdot y) \cdot x^{\rho} = y$$

holds with $x^{\lambda} = e/x$, $x^{\rho} = x \backslash e$.

Proposition

For a Steiner-like loop L the following properties are equivalent:
a) L is right alternative,
b) L satisfies the right inverse property,
c) the set $\mathcal{F} = \{f(z,z);\ z \in S\}$ *is contained in the centre of A and for all* $x, y \in S \setminus \{e\}$ *with* $x \neq y$ *the identity* $h(x)h(y)h(x)h(xy) = f(x,x)$ *holds such that xy is the third element of the block of* $S \setminus \{e\}$ *determined by x, y.*

Theorem

Let L be a Steiner-like loop. We assume that the set $\mathcal{F} = \{f(x,x); x \in S\}$ *is contained in the centre* $Z(A)$ *of the group* A. *Let D be the group generated by the set* $\mathcal{H} = \{h(z);\ z \in S \setminus \{e\}\}$. *If S has more than* 4 *elements, then the following properties are equivalent:*

a) For all $x, y \in S \setminus \{e\}$ *with* $x \neq y$ *the identity*

$$h(x)h(y)h(x)h(xy) = f(x,x) \qquad (3)$$

holds.

Theorem

b) If the group D is abelian, then precisely one of the following cases holds:
(i) For all $x \in S \setminus \{e\}$ one has $h(x) = t$, $f(x,x) = t^4$ and $D = \langle t \rangle$.
(ii) There is a subloop U of the Steiner loop S such that S is the direct product of U and $\mathbb{Z}_2$ and $h(x) = t \neq 1$ for all $x \in U$ whereas $h(y) = t\omega = \omega t$ with a fixed involution ω for all $y \in S \setminus U$. The elements $f(x,x)$ have the form t^4 or $t^4\omega$ depending whether $x \in U$, respectively $x \notin U$. The group D is the direct product of $\langle t \rangle$ and $\langle \omega \rangle$.

Theorem

If the group D is non-abelian and the Steiner loop S is finite, then one has $h(x) = u\omega_x = \omega_x u$, where u is a fixed element of A which centralizes any element of D and for the order $o(\omega_x)$ of the element ω_x of A one has $o(\omega_x) \leq 2$. For every $x \in S \setminus \{e\}$ one has $f(x,x) = u^4$. If Γ is the set of different involutions in the factor group $D\langle u\rangle/\langle u\rangle = \langle \omega_x\langle u\rangle, x \in S \setminus \{e\}\rangle$, then $D\langle u\rangle/\langle u\rangle$ is a restricted Fischer group generated by Γ. If for the mapping $h : x \mapsto u\omega_x$ one has $\omega_x\omega_y \neq \omega_y\omega_x$ for all $x, y \in S \setminus \{e\}$, $x \neq y$, then $S \setminus \{e\}$ is a Hall system.
The Steiner-like loop L is the direct product of A and S precisely if the group D has order ≤ 2.

The group D generated by the set $\{h(x); x \in S \setminus \{e\}\}$ is abelian except the loops satisfying only the right alternative or the right inverse property.

Definition

A (restricted) Fischer group is a pair (G, E) consisting of a group G and a system of generators $E \subset G$ satisfying the following conditions:
(i) For all $x \in E$ one has $x^2 = 1$.
(ii) For all $x, y \in E$ we have $(xy)^3 = 1$ and $xyx \in E$.
A Steiner triple system $S \setminus \{e\}$ is a Hall system if any three non-collinear points of $S \setminus \{e\}$ generates the affine plane of order 3.

Proposition

Let L be a Steiner-like loop such that $S = \{e, x, y, xy\}$ *is the elementary abelian group of order* 4*. We assume that the set* $\mathcal{F} = \{f(x,x); x \in S\}$ *is contained in the centre* $Z(A)$ *of the group* A*. Let* D *be the group generated by the set* $\mathcal{H} = \{h(z);\ z \in S \setminus \{e\}\}$*. Then the following properties are equivalent:*

a) For all $x, y \in S \setminus \{e\}$ *with* $x \neq y$ *the identity*

$$h(x)h(y)h(x)h(xy) = f(x,x)$$

holds.

Proposition

b) If the group D is non-abelian, then it is a product $K \cdot \langle a \rangle$, where $a = h(x)$ and $t = h(x)h(y)$, such that the abelian group K has the form $\langle s \rangle \times Z(D)$, where the cyclic group $\langle s \rangle$ of order 3 is the commutator subgroup D' of D, the group $Z(D)$ is generated by a^2 and t^3 and a inverts any element of $\langle s \rangle$.
If the group D is abelian, then D is generated by three elements $h(x) = a$, $h(y) = b$ and $h(xy) = c$ such that a, b and c commute and one has

$$f(x,x) = al, \; f(y,y) = bl, \; f(xy,xy) = cl,$$

where $l = abc$.

Proposition

For a Steiner-like loop L the following properties are equivalent:
a) L is left alternative,
b) L satisfies the left inverse property,
c) the sets $\mathcal{F} = \{f(z,z);\ z \in S\}$ *and* $\mathcal{K} = \{h(x)h(y);\ x, y \in S \setminus \{e\}, x \neq y\}$ *are contained in the centre of A and for all* $x, y \in S \setminus \{e\}$ *with* $x \neq y$ *the identity* $h(x)h(y)h(x)h(xy) = f(x,x)$ *holds.*

Proposition

A Steiner-like loop L satisfies the cross inverse property if and only if the group A is commutative and for all $x, y \in S \setminus \{e\}$ *with* $x \neq y$ *the identity* $h(x)h(y)h(x)h(xy) = f(x,x)$ *holds.*

Let $x^{-1} := x^{\lambda} = x^{\rho}$. A loop L has the automorphic inverse property if the identity $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ holds for all $x, y \in L$.

Proposition

A Steiner-like loop L has the automorphic inverse property if and only if the group A is commutative and for all $x, y \in S \setminus \{e\}$ and $x \neq y$ the identity

$$h(x)^2 h(y)^2 = f(x,x) f(y,y) f(xy,xy)^{-1}$$

holds.

Proposition

Let S be a Steiner loop and A be an abelian group. Let f be a mapping $S \times S \to A$ such that for $x \neq y \in S \setminus \{e\}$ one has $f(x,y) = h(x)h(y)$ for a mapping $h : S \setminus \{e\} \to A$ and $f(x,e) = f(e,x) = 1$ for all $x \in S$. Then the following assertions are equivalent:

a) For all $x, y \in S \setminus \{e\}$ and $x \neq y$ the identity

$$h(x)^2 h(y)^2 = f(x,x)f(y,y)f(xy,xy)^{-1}$$

holds.

b) If S has more than 4 elements, then the elements $h(x)$ have the form $h(x) = u\omega_x$, where u is a fixed element of A and for the order $o(\omega_x)$ of the element ω_x one has $o(\omega_x) \leq 2$.

The elements $f(x,x)$ have the form $f(x,x) = u^4\rho_x$ such that for the order $o(\rho_x)$ of the element ρ_x one has $o(\rho_x) \leq 2$ and $\rho_x\rho_y = \rho_{xy}$ for all $x \neq y \in S \setminus \{e\}$.

Proposition

If $S = \{e, x, y, xy\}$ is the elementary abelian group of order 4, then one has

$$h(x) = \alpha, \quad h(y) = \beta, \quad h(xy) = \gamma,$$

$$f(x,x) = \alpha^2\beta\gamma\rho_x, \quad f(y,y) = \alpha\beta^2\gamma\rho_y,$$

$$f(xy,xy) = \alpha\beta\gamma^2\rho_x\rho_y,$$

where α, β, γ are elements of A, whereas the orders of ρ_x, ρ_y are at most 2.

If the Steiner loop S has more than four elements, then the Steiner-like loop L is the direct product of A and S precisely if the group D is the direct product $\langle u \rangle \times \langle \omega \rangle$ such that for the orders $o(u)$ as well as $o(\omega)$ one has $o(u) \leq 2$ and $o(\omega) \leq 2$ and $\rho_x = 1$ for all $x \in S \setminus \{e\}$. If $S = \{e, x, y, xy\}$ is the elementary abelian group of order 4, then L is the direct product of A and S precisely if the group D has order ≤ 2 and $\rho_x = 1$ for all $x \in S \setminus \{e\}$.

Definition

A loop L is a right Bol loop if $z \cdot [(x \cdot y) \cdot x] = [(z \cdot x) \cdot y] \cdot x$ *holds for all* $x, y, z \in L$.

Proposition

Let L be a proper Steiner-like loop.
a) If $S = \{e, x, y, xy\}$ *is the elementary abelian group of order* 4*, then the loop L satisfies the right Bol identity if and only if the range of h is commutative, the set* $\{f(x,x);\ x \in S \setminus \{e\}\}$ *is contained in the centre of A and identity* $h(x)h(y)h(x)h(xy) = f(x,x)$ *holds. b) If the Steiner loop S has more than* 4 *elements, then the loop L is a right Bol loop precisely if S is an elementary abelian* 2*-group and for all* $x \in S \setminus \{e\}$ *one has* $h(x) = t$ *and* $f(x,x) = t^4$ *with a fixed element t of A, but* $t^2 \notin Z(A)$.

Definition

A loop L is a left Bol loop if
$[x \cdot (y \cdot x)] \cdot z = x \cdot [y \cdot (x \cdot z)]$ *holds for all* $x, y, z \in L$.

Theorem

Let L be a Steiner-like loop such that S has more than 4 elements. Then the following conditions are equivalent:
(i) L is a group.
(ii) S is an elementary abelian 2-group and for all $x \in S \setminus \{e\}$ *one has* $h(x) = t$ *and* $f(x, x) = t^4$*, where t is a fixed element of A such that* $t^2 \in Z(A)$.
(iii) L is a left Bol loop.

A Steiner-like right Bol loop L does not need to be left alternative as the following examples show. Let A be a finite simple group which is not isomorphic to one of the following groups: $PSL(2, 2^n)$, $n > 1$, $PSL(2, q)$, $q \equiv 3$ or $5 \pmod 8$, $q > 3$, the Janko group of order 175560, a group of Ree type of order $q^3(q-1)(q^3+1)$, where $q = 3^{2k+1}$, $k \geq 1$. Then A contains an element u of order 4. Putting $h(x) = u$ for all $x \in S \setminus \{e\}$ we obtain a right Bol loop L which is not left alternative and has exponent 4 since $f(x, x) = 1$.

Theorem

Let L be a Steiner-like loop such that $S = \{e, x, y, xy\}$ *is the elementary abelian group of order* 4*. Then the following conditions are equivalent:*
(i) L is a group.
(ii) L is left alternative or it has the left inverse property.
(iii) L is a left Bol loop.
Moreover, L is an abelian group if and only if L has the cross inverse property.

Definition

A loop L satisfies the weak inverse property if for $x, y, z \in L$ one has

$$x \cdot (y \cdot z) = e \quad \textit{whenever} \quad (x \cdot y) \cdot z = e.$$

Proposition

A Steiner-like loop L has the weak inverse property if and only if the set $\{f(x, y);\ x, y \in S\}$ is contained in the centre of A and for all $x \in S \setminus \{e\}$ one has $f(x, x) = sh(x) = h(x)s$, where s is a fixed element of A.

Let L be a Steiner-like loop. A left translation $\lambda_{(a,\alpha)} : L \to L$ is the bijection

$$(x, \xi) \longmapsto (a, \alpha)(x, \xi) = (ax, f(a, x)\alpha\xi)$$

and a right translation $\rho_{(a,\alpha)} : L \to L$ is the bijection

$$(x, \xi) \longmapsto (x, \xi)(a, \alpha) = (ax, f(x, a)\xi\alpha).$$

Proposition

Let L be a Steiner-like loop. Then the group G_r generated by all right translations of L is an extension of A by the group Σ generated by the translations of the set $\{\rho_{(a,1)};\ a \in S\}$.

To obtain an analogous result for the group G_l generated by all left translations of L we must suppose that the set $\{f(x,y);\ x,y \in S\}$ is contained in the centre of A.

Proposition

Let L be a Steiner-like loop. We assume that L has one of the following properties:
a) L is flexible,
b) L is left alternative or it has the left inverse property,
c) L has the cross inverse property,
d) L has the automorphic inverse property,
e) L has the weak inverse property.
Then the group G_l generated by all left translations of L is an extension of A by the group Σ' generated by the translations of the set $\{\lambda_{(a,1)};\ a \in S\}$.

Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops of type (1) realized on the set $S \times A$ by the multiplications

$$(x, \xi) \cdot (y, \eta) = (xy, f_1(x, y)\xi\eta)$$

respectively

$$(x, \xi) * (y, \eta) = (xy, f_2(x, y)\xi\eta).$$

We consider isomorphism $\alpha : L_1 \to L_2$ such that $(e, \xi)^\alpha = (e, \xi^{\alpha''})$, where α'' is an automorphism of A. For such an isomorphism $\alpha : L_1 \to L_2$ one has $(x, 1)^\alpha = (x^{\alpha'}, \rho(x^{\alpha'}))$, where ρ is a map from S onto A.

Proposition

Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops of type (1) belonging to the functions f_1, respectively f_2. The map

$$\beta : L_1 \to L_2, (x, \xi) \mapsto (x^{\alpha'}, \rho(x^{\alpha'})\xi^{\alpha''})$$

defines an isomorphism of L_1 onto L_2 if and only if α' is an automorphism of the loop S, α'' is an automorphism of A, for the map $\rho : S \to A$ one has the set $\{\rho(x^{\alpha'});\ x \in S, \alpha' \in Aut(S)\}$ is contained in the centre $Z(A)$ of A and for all $x, y \in S$ identity

$$f_1(x, y)^{\alpha''} = \rho((xy)^{\alpha'})^{-1}\rho(x^{\alpha'})\rho(y^{\alpha'})f_2(x^{\alpha'}, y^{\alpha'})$$

holds.

Proposition

Let L_1 and L_2 be two Steiner-like loops with respect to the functions f_1, respectively f_2. The map $\beta : L_1 \to L_2; (x, \xi) \mapsto (x^{\alpha'}, \rho(x^{\alpha'})\xi^{\alpha''})$ is an isomorphism between these Steiner-like loops if and only if α' is an automorphism of the Steiner loop S, α'' is an automorphism of A, the set $\{\rho(x^{\alpha'});\ x \in S, \alpha' \in Aut(S)\}$ is contained in the centre of A, for all $x \in S$ identities

$$f_1(x,x)^{\alpha''} f_2(x^{\alpha'}, x^{\alpha'})^{-1} = \rho(x^{\alpha'})^2$$

and for all $x, y \in S \setminus \{e\}$, $x \neq y$ identity

$$h_1(x)^{\alpha''} h_1(y)^{\alpha''} = \rho((xy)^{\alpha'})^{-1} \rho(x^{\alpha'})\rho(y^{\alpha'}) h_2(x^{\alpha'}) h_2(y^{\alpha'})$$

are satisfied.

Proposition

Let L be a loop of type (1). A mapping $\alpha : L \to L$ defined by $(x, \xi)^{\alpha} = (x^{\alpha'}, \rho(x^{\alpha'})\xi^{\alpha''})$ is an automorphism of L if and only if α' is an automorphism of the loop S, α'' is an automorphism of the group A, for the map $\rho : S \to A$ with $\rho(e) = 1$ one has the set $\{\rho(x^{\alpha'});\ x \in S, \alpha' \in Aut(S)\}$ is contained in the centre $Z(A)$ of A and for all $x, y \in S$ identity

$$f(x,y)^{\alpha''} = \rho((xy)^{\alpha'})^{-1}\rho(x^{\alpha'})\rho(y^{\alpha'})f(x^{\alpha'}, y^{\alpha'})$$

holds.

The map $\rho : S \to Z(A)$, $x \mapsto \rho(x)$ is a homomorphism from S into $Z(A)$ if and only if there are $\alpha' \in Aut(S)$, $\alpha'' \in Aut(A)$ such that $f(x,y)^{\alpha''} = f(x^{\alpha'}, y^{\alpha'})$ for all $x, y \in S$. Any homomorphism ρ from S into the centre $Z(A)$ determines an automorphism of L which is the mapping $\beta_\rho : (x,\xi) \mapsto (x, \rho(x)\xi)$. The set of these automorphisms forms the normal subgroup Ψ of the automorphism group Γ of L which consists of automorphisms inducing on S as well as on A the identity. The group Ψ is commutative.

Proposition

Let L be a Steiner-like loop. The map $\beta : L \to L$ *defined by* $(x,\xi)^\beta = (x^{\alpha'}, \rho(x^{\alpha'})\xi^{\alpha''})$ *is an automorphism of L if and only if for all* $x \in S$ *identity*

$$\rho(x^{\alpha'})^2 = f(x,x)^{\alpha''} f(x^{\alpha'}, x^{\alpha'})^{-1}$$

and for all $x, y \in S \setminus \{e\}$, $x \neq y$ *identity*

$$\rho(x^{\alpha'})\rho(y^{\alpha'})h(x^{\alpha'})h(y^{\alpha'}) = \rho((xy)^{\alpha'})h(x)^{\alpha''}h(y)^{\alpha''}$$

hold.
The normal subgroup Ψ *of the automorphism group* Γ *of L is an elementary abelian 2-group.*

A loop extension of type (1) with respect to the factor system f may be also defined in such a way that the multiplication on the set $S \times A$ is given by

$$(x, \xi)(y, \eta) = (xy, \xi f(x, y)\eta),$$

respectively

$$(x, \xi)(y, \eta) = (xy, \xi\eta f(x, y)).$$

This yields loops L^*, respectively L^{**} which coincide with the extension given by multiplication (1) if and only if the set $\{f(x, y);\ x, y \in S\}$ is contained in the centre of A.

Proposition

A Steiner-like loop L^{**} is left alternative or it satisfies the left inverse property precisely if the set $\mathcal{F} = \{f(z,z);\ z \in S\}$ is contained in the centre of A and identity $h(x)h(y)h(x)h(xy) = f(x,x)$ holds.

Proposition

Let L^{**} be a proper Steiner-like loop.
a) If $S = \{e, x, y, xy\}$ is the elementary abelian group of order 4, then the loop L^{**} satisfies the left Bol identity if and only if the range of h is commutative, the set $\{f(x,x);\ x \in S\}$ is contained in the centre of A and identity $h(x)h(y)h(x)h(xy) = f(x,x)$ holds.
b) If the Steiner loop S has more than 4 elements, then the loop L^{**} is a left Bol loop precisely if S is an elementary abelian 2-group and for all $x \in S \setminus \{e\}$ one has $h(x) = t$ and $f(x,x) = t^4$ with a fixed element t of A, but $t^2 \notin Z(A)$.

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