THE MINIMAL BASE SIZE FOR A \( p \)-SOLVABLE LINEAR GROUP

ZOLTÁN HALASI AND ATTILA MARÓTI

Dedicated to the memory of Akos Seress.

Abstract. Let \( V \) be a finite vector space over a finite field of order \( q \) and of characteristic \( p \). Let \( G \leq GL(V) \) be a \( p \)-solvable completely reducible linear group. Then there exists a base for \( G \) on \( V \) of size at most 2 unless \( q \leq 4 \) in which case there exists a base of size at most 3. This extends a recent result of Halasi and Podoski and generalizes a theorem of Seress. A generalization of a theorem of Palfy and Wolf is also given.

1. Introduction

For a finite permutation group \( H \leq \text{Sym}(\Omega) \), a subset of the finite set \( \Omega \) is called a base, if its pointwise stabilizer in \( H \) is the identity. The minimal base size of \( H \) (on \( \Omega \)) is denoted by \( b(H) \). Notice that \( |H| \leq |\Omega|^b(H) \).

One of the highlights of the vast literature on base sizes of permutation groups is the celebrated paper of Á. Seress [21] in which it is proved that \( b(H) \leq 4 \) whenever \( H \) is a solvable primitive permutation group. Since a solvable primitive permutation group is of affine type, this result is equivalent to saying that a solvable irreducible linear subgroup \( G \) of \( GL(V) \) has a base of size at most 3 (in its natural action on \( V \)) where \( V \) is a finite vector space.

There are a number of results on base sizes of linear groups. For example, D. Gluck and K. Magaard [10, Corollary 3.3] have shown that a subgroup \( G \) of \( GL(V) \) with \((|G|, |V|) = 1 \) admits a base of size at most 94. If in addition it is assumed that \( G \) is supersolvable or of odd order then \( b(G) \leq 2 \) by results of T. R. Wolf [24, Theorem A] and S. Dolfi [4, Theorem 1.3]. Later Dolfi [5, Theorem 1.1] and E. P. Vdovin [22, Theorem 1.1] generalized this result to solvable coprime linear groups. Finally, Z. Halasi and K. Podoski [12, Theorem 1.1] improved this result significantly, by proving that even the solvability assumption can be dropped, and \( b(G) \leq 2 \) for any coprime linear group \( G \).

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We note that for a solvable subgroup $G$ of $GL(V)$ acting completely reducibly on $V$ we have $b(G) \leq 2$ if the Sylow 2-subgroups of $GV$ are Abelian (see [6, Theorem 2]) or if $|G|$ is not divisible by 3 (see [25, Theorem 2.3]).

The following definition has been introduced by M. W. Liebeck and A. Shalev in [16]. For a linear group $G \leq GL(V)$ we say that $\{v_1, \ldots, v_k\} \subseteq V$ is a strong base for $G$ if any element of $G$ fixing $\langle v_i \rangle$ for every $1 \leq i \leq k$ is a scalar transformation. The minimal size of a strong base for $G$ is denoted by $b^*(G)$. It is known that $b(G) \leq b^*(G) \leq b(G) + 1$ (see [16, Lemma 3.1]). Furthermore, also $b^*(G) \leq 2$ holds for coprime linear groups by [12, Lemma 3.3 and Theorem 1.1].

The following theorem generalizes the above-mentioned result of Seress [21] and extends that of Halasi and Podoski [12] to $p$-solvable groups.

**Theorem 1.1.** Let $V$ be a finite vector space over a field of order $q$ and of characteristic $p$. If $G \leq GL(V)$ is a $p$-solvable group acting completely reducibly on $V$, then $b^*(G) \leq 2$ unless $q \leq 4$. Moreover if $q \leq 4$ then $b^*(G) \leq 3$.

One of the motivations of Seress [21] was a famous result of P. P. Pálfy [19, Theorem 1] and Wolf [23, Theorem 3.1] stating that a solvable primitive permutation group of degree $n$ has order at most $24^{-1/3}n^d$ where $d = 1 + \log_q(48 \cdot 24^{1/3}) = 3.243\ldots$, that is to say, a solvable irreducible subgroup $G$ of $GL(V)$ has size at most $24^{-1/3}|V|^{d-1}$. See also the book by Manz and Wolf [17, Chap. 1, §3, Thm. 3.5.A]. (This bound is attained for infinitely many groups.) In the following we generalize this result to $p$-solvable linear groups $G$.

**Theorem 1.2.** Let $V$ be a finite vector space over a field of characteristic $p$. If $G \leq GL(V)$ is a $p$-solvable group acting completely reducibly on $V$, then $|G| \leq 24^{-1/3}|V|^{d-1}$ where $d$ is as above.

We note that the bounds in Theorem 1.1 are best possible for all values of $q$. Indeed, there are infinitely many irreducible solvable linear groups $G \leq GL(V)$ with $|G| > |V|^2$ for $q = 2$ or 3 (see [19, Theorem 1] or [23, Proposition 3.2]) and there are even infinitely many odd order completely reducible linear groups $G \leq GL(V)$ with $|G| > |V|$ for $q \geq 5$ (see [20, Theorem 3B] and the remark that follows). For $q = 4$ we note that [8] shows that there are primitive, irreducible solvable linear subgroups $H$ of $GL(3,4)$ with $b(H) = 3$ and thus there are infinitely many imprimitive, irreducible solvable linear groups $G = H \wr S \leq GL(3r,4)$ with $b(G) = 3$ where $S$ is a solvable transitive permutation group of degree $r$.

Theorem 1.1 has been applied in [2] to Gluck's conjecture.

2. Preliminaries

Throughout this paper let $\mathbb{F}_q$ be a finite field of characteristic $p$ and let $V$ be an $n$-dimensional vector space over $\mathbb{F}_q$. Furthermore, let $G \leq GL(V)$ be a linear group acting on $V$ in the natural way, let $b(G)$ denote its minimal base size, and let $b^*(G)$ denote its minimal strong base size (both notions defined in Section 1).

If the vector space $V$ is fixed, then the group of scalar transformations of $V$ (the center of $GL(V)$) will be denoted by $Z$. Thus $Z \simeq \mathbb{F}_q^\times$, the multiplicative group of the base field. As $G \leq GL(V)$ is $p$-solvable if and only if $GZ$ is $p$-solvable, we can (and we will) always assume, in the proofs of Theorems 1.1 and 1.2, that $G$ contains $Z$. After choosing a basis $\{v_1, \ldots, v_n\} \subseteq V$, we will always identify the group $GL(V)$ with the group $GL(n,q)$. 
Put \( t(q) = 3 \) for \( q \leq 4 \) and \( t(q) = 2 \) for \( q \geq 5 \).

Finally, if \( G \leq GL(V) \) and \( X \subseteq V \), then \( C_G(X) = \{ g \in G \mid g(x) = x \ \forall x \in X \} \) and \( N_G(X) = \{ g \in G \mid g(x) \in X \ \forall x \in X \} \) will denote the pointwise and setwise stabilizer of \( X \) in \( G \), respectively.

3. Special bases in linear groups

In this section we will show that there exist bases of special kinds for certain linear groups. As a consequence (Corollary 3.3), we derive that it is sufficient to establish the required bounds in Theorem 1.1 for \( b(G) \) rather than for \( b^*(G) \).

**Theorem 3.1.** Let \( V \) be an \( n \)-dimensional vector space over \( \mathbb{F}_q \), a field of characteristic \( p \) and let \( Z \leq G \leq GL(V) \) be a \( p \)-solvable linear group.

1. If \( n = 2 \) and \( q \geq 5 \), then at least one of the following holds.
   a. There is a basis \( x, y \in V \) such that \( N_G(\{x\}) \subseteq N_G(\{y\}) \).
   b. \( p = 2 \) and there is a basis \( x, y \in V \) such that \( N_G(\{x\}) = Z \times C_2 \) and the involution \( g \in N_G(\{x\}) \) satisfies \( g(x) = x \) and \( g(y) = y + x \).

2. If \( n = 3 \) and \( q = 3 \) or 4, then at least one of the following holds.
   a. There is a basis \( x, y, z \in V \) such that \( N_G(\{x\}) \cap N_G(\{y\}) \subseteq N_G(\{z\}) \).
   b. There is a basis \( x, y, z \in V \) such that \( N_G(\{y, z\}) = G \).

*Proof.* If \( G \leq GL(V) \) leaves invariant a 1-dimensional subspace of \( V \), then \( 1/(a) \) or \( 2/(a) \) is satisfied. If \( n = 3 \) and \( G \) leaves invariant a 2-dimensional subspace of \( V \) then \( 2/(b) \) is satisfied. Thus we may assume that \( G \) acts irreducibly on \( V \).

If \( G \) acts imprimitively on \( V \) then it embeds in \( G_{n-1} \wr S_n \) where the base group acts on \( \langle x \rangle \oplus \langle y \rangle \) if \( n = 2 \) and on \( \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle \) if \( n = 3 \), for some vectors \( x, y, z \in V \). In the first case \( N_G(\{x\}) \) is diagonalizable and thus \( 1/(a) \) is satisfied, while in the second case \( N_G(\{x\}) \cap N_G(\{y\}) \) is diagonalizable and thus \( 2/(a) \) is satisfied. We may thus assume that \( G \) acts primitively (and irreducibly) on \( V \).

Since \( G \) is \( p \)-solvable by assumption, we see that \( G \) does not contain \( SL(V) \).

First consider statement (1). By considering the action of \( G \) on the set \( S \) of \( 1 \)-dimensional subspaces of \( V \), we may assume that the number of Sylow \( p \)-subgroups of \( G \) is equal to \( |S| = q + 1 \). For otherwise there exists \( x \in S \) whose stabilizer in \( G \) is a \( p' \)-group and thus Maschke’s theorem gives \( 1/(a) \). For \( q = p \) any subgroup of \( GL(V) \) with \( q + 1 \) Sylow \( p \)-subgroups contains \( SL(V) \), so in this case we are done. So assume that \( q > p \).

Since \( G \) acts transitively on the set of Sylow \( p \)-subgroups of \( G \) and every Sylow \( p \)-subgroup stabilizes a unique subspace in \( S \), it follows that \( G \) acts transitively on \( S \). Moreover since \( Z \leq G \) it also follows that \( G \) acts transitively on the set of non-zero vectors of \( V \).

By Hering’s theorem (see [13, Chapter XII, Remark 7.5 (a)]) we see that if \( q \) is odd (and not a prime by assumption) then \( q \) must be 9 and \( G \) has a normal subgroup isomorphic to \( SL(2,5) \) (case (5)). But then \( G \) is not 3-solvable and so we can rule out this possibility. Similarly, if \( q \) is even, then the only possibility is that \( G \geq Z \) normalizes a Singer cycle \( GL(1, q^2) \) (case (1)). The only such group not satisfying \( 1/(a) \) is the full semilinear group \( \Gamma(1, q^2) \simeq GL(1, q^2).2 \). In this case taking \( x \) to be any non-zero vector in \( V \) we have \( N_G(\langle x \rangle) = Z \times C_2 \) and the involution \( g \) in \( N_G(\langle x \rangle) \) satisfies \( g(x) = x \) and \( g(y) = y + x \) for some \( y \in V \).

Finally, statement (2) has been checked with [8] by using the list of all primitive permutation groups of degrees 27 and 64, respectively. \( \square \)
As a direct consequence we get the following.

**Corollary 3.2.** Let us assume that $Z \leq G \leq GL(V)$ is a $p$-solvable linear group with $b(G) \leq t(q)$.

(1) If $q \geq 5$, then one of the following holds.
   (a) There exists a base $x, y \in V$ such that $N_G(\langle x, y \rangle) \subseteq N_G(\langle y \rangle)$.
   (b) $p = 2$ and there exists a base $x, y \in V$ such that any non-identity element of $C_G(x) \cap N_G(\langle x, y \rangle)$ takes $y$ to $y + x$.

(2) If $q \leq 4$, then at least one of the following holds.
   (a) There exists a base $x, y, z \in V$ such that $N_G(\langle x, y, z \rangle) \subseteq N_G(\langle y, z \rangle)$.
   (b) There exists a base $x, y, z \in V$ such that $N_G(\langle x, y, z \rangle) \subseteq N_G(\langle y, z \rangle)$ with $x \notin \langle y, z \rangle$.

**Proof.** First, 1/(a) or 2/(a) holds if $\dim(V) < t(q)$ so assume that $\dim(V) \geq t(q)$. Both parts of the corollary can be proved by choosing a subspace $U \leq V$ of dimension $t(q)$ generated by a base for $G$ and by restricting $N_G(U)$ to this subspace. Notice that the image of this restriction is also $p$-solvable, so Theorem 3.1 can be applied.

**Corollary 3.3.** Let $V$ be a vector space over the field $\mathbb{F}_q$ of characteristic $p$. Let $Z \leq G \leq GL(V)$ be $p$-solvable with $b(G) \leq t(q)$. Then $b^*(G) \leq t(q)$.

**Proof.** We may assume that $\dim(V) \geq t(q)$ and that $q > 2$. Let us choose a base for $G$ of size $t(q)$ satisfying the property given in Corollary 3.2. For $q \geq 5$, if $x, y \in V$ is such a base, then $x, x + y$ is a strong base for $G$. Likewise, for $q = 3$ or 4, if $x, y, z \in V$ is a base satisfying (2/a) of Corollary 3.2, then $x, y, x + y + z$ is a strong base for $G$. Finally, in case $x, y, z \in V$ is a base for $G$ satisfying (2/b) of Corollary 3.2, then $x, y + x, z + x$ is a strong base for $G$.

4. FURTHER REDUCTIONS

Let us use induction on the dimension $n$ of $V$ in the proofs of Theorems 1.1 and 1.2. The case $n = 1$ is clear. Let us assume that $n > 1$ and that both Theorems 1.1 and 1.2 are true for dimensions less than $n$.

First we reduce the proof of both theorems for the case when $G \leq GL(V)$ acts irreducibly on $V$. For otherwise let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be a decomposition of $V$ to irreducible $\mathbb{F}_q G$-modules.

By induction, there exist vectors $x_{i,1}, \ldots, x_{i,t(q)}$ in $V_i$ for $1 \leq i \leq k$ with the property that $C_G(\{x_{i,1}, \ldots, x_{i,t(q)}\})$ is precisely the kernel of the action of $G$ on $V_i$. Now put $x_j = \sum_{i=1}^{k} x_{i,j}$ for $1 \leq j \leq t(q)$. One can see that $C_G(\{x_{1}, \ldots, x_{t(q)}\}) = \cap_{i=1}^{k} C_G(V_i) = 1$.

For Theorem 1.2 notice that $G$ is a subgroup of a direct product $\times_{i=1}^{k} H_i$ of $p$-solvable groups $H_i$ acting irreducibly and faithfully on the $V_i$’s. Hence we have

$$|G| \leq \prod_{i=1}^{k} |H_i| \leq \prod_{i=1}^{k} \left(24^{-1/3} |V_i|^{d-1}\right) = 24^{-k/3} |V|^{d-1}$$

by induction.

So from now on we will assume that $G \leq GL(V)$ acts irreducibly on $V.$
For Theorem 1.1 we may also assume that \( q \neq 2, 4 \). Otherwise, \( G \) is solvable by the Odd Order Theorem and we can use the result of Seress [21].

For Theorem 1.2 we may assume that \(|G| > |V|^2\). If \(|G| \leq |V|^2\) then \(|V|^2 < 24^{-1/2}|V|^{d-1}\) for \(|V| \geq 79\), so we may assume that \(|V| \leq 73\). If \(|V|\) is a prime or \( p \geq 2 \) then \( G \) is solvable and the theorem of Pálfy [19] and Wolf [23] can be applied. Hence the cases \(|V| = 5^2, 7^2, 3^2\) or \( 3^3 \) remain to be examined. But in these cases there is no non-solvable, \( p \)-solvable irreducible subgroup of \( GL(V) \) (see [8]).

Now, if \( b(G) \leq 2 \) then \(|G| \leq |V|^2\). So, once Theorem 1.1 is proved, it remains to prove Theorem 1.2 only in case \( q = 3 \) and \( b(G) > 2 \).

5. IMPRIMITIVE LINEAR GROUPS

In this section we show that we may assume (for the proofs of Theorems 1.1 and 1.2) that \( G \) is a primitive (irreducible) subgroup of \( GL(V) \).

We first consider Theorem 1.1.

For \( G \leq GL(V) \) an irreducible imprimitive linear group, let \( V = V_1 \oplus \cdots \oplus V_k \) be a decomposition of \( V \) into subspaces such that \( G \) permutes these subspaces in a transitive and primitive way. This action of \( G \) defines a homomorphism from \( G \) into the symmetric group \( Sym(\Omega) \) for \( \Omega = \{V_1, \ldots, V_k\} \) with kernel \( N \).

The factor group \( G/N \leq S_k \) is \( p \)-solvable, so it does not involve \( A_q \) for \( q \geq 5 \) and it does not involve \( A_5 \) for \( q = 3 \). By using [12, Theorem 2.3] it follows that for \( q \geq 5 \) there is a vector \( a = (a_1, \ldots, a_k) \in \mathbb{F}_q^k \) such that \( C_{G/N}(a) = 1 \). (Here, \( G/N \) acts on \( \mathbb{F}_q^k \) by permuting coordinates.) If \( q = 3 \) then again by [12, Theorem 2.3] we know that there is a 5 (and thus 9) part partition of \( \Omega \) whose stabilizer in \( G/N \) is trivial. This implies that, for \( q = 3 \), there is a pair of vectors \( a = (a_1, \ldots, a_k) \), \( b = (b_1, \ldots, b_k) \in \mathbb{F}_3^k \) such that \( C_{G/N}(a) \cap C_{G/N}(b) = 1 \).

In fact for \( q \geq 8 \) even we can say a bit more. For such a \( q \) let \( S \) be a subset of \( \mathbb{F}_q \) of size \( q/2 \) with the property that for each \( c \in \mathbb{F}_q \) exactly one of \( c \) and \( c + 1 \) is contained in \( S \). By [3, Lemma 1 (c)], there is a \( 4 \leq q/2 \) part partition of \( \Omega \) whose stabilizer in \( G \) is \( N \), so there exists a vector \( a = (a_1, \ldots, a_k) \in S_k \) such that \( C_{G/N}(a) = 1 \). (Actually, in our case, this already follows from [9, Theorem 1] by noting that since \( q \) is even, \( p = 2 \), and thus \( G/N \) is a solvable primitive permutation group.)

For each \( 1 \leq i \leq k \) let \( H_i = N_G(V_i) \), so \( N = \cap_i H_i \). By induction (on the dimension), there is a base in \( V_1 \) of size \( t(q) \) for \( H_1/C_{H_1}(V_1) \).

Now we can use Corollary 3.2. First let \( q \geq 5 \). Then there is a base \( x_1, y_1 \in V_1 \) for \( K_1 = H_1/C_{H_1}(V_1) \leq GL(V_1) \) such that \( N_{K_1}(\langle x_1 \rangle) \cap N_{K_1}(\langle y_1 \rangle) \subseteq N_{K_1}(\langle y_1 \rangle) \) or that any non-identity element of \( C_{K_1}(x_1) \cap N_{K_1}(\langle x_1, y_1 \rangle) \) takes \( y_1 \) to \( y_1 + x_1 \).

Let \( \{g_1 = 1, g_2, \ldots, g_k\} \) be a set of left coset representatives for \( H_1 \) in \( G \) and \( x_i = g_i x_1, \ y_i = g_i y_1 \) for every \( i \). Now let

\[
x = \sum_{i=1}^k x_i, \quad y = \sum_{i=1}^k y_i + a_i x_i.
\]

In case \( q = 3 \) let \( x_1, y_1, z_1 \in V_1 \) be a base for \( K_1 = H_1/C_{H_1}(V_1) \leq GL(V_1) \) satisfying (2/a) or (2/b) of Corollary 3.2. Again, let \( \{g_1 = 1, g_2, \ldots, g_k\} \) be a set of left coset representatives for \( H_1 \) in \( G \) and \( x_i = g_i x_1, \ y_i = g_i y_1, \ z_i = g_i z_1 \) for every
i. Depending on which part of part (2) of Corollary 3.2 is satisfied for \( x_1, y_1, z_1 \) let

\[
x = \sum_{i=1}^{k} x_i, \quad y = \sum_{i=1}^{k} y_i, \quad z = \sum_{i=1}^{k} (z_i + b_i x_i + a_i y_i)
\]

if (2/a) holds,

\[
x = \sum_{i=1}^{k} x_i, \quad y = \sum_{i=1}^{k} (y_i + a_i x_i), \quad z = \sum_{i=1}^{k} (z_i + b_i x_i)
\]

if (2/b) holds.

In each case, it is easy to see that the given set of vectors is a base for \( G \) by using similar arguments as in the proof of [12, Theorem 2.6]. For the convenience of the reader, we present a proof here for the case (2/a).

Let \( x, y, z \) given as above and \( g \in C_G(x) \cap C_G(y) \cap C_G(z) \). Furthermore, let \( \sigma \in S_k \) be the permutation associated to the action of \( g \) on \( \Omega = \{V_1, \ldots, V_k\} \). Then \( g(x) = x \), \( g(y) = y \) implies that \( g(x_i) = x_{\sigma(i)} \), \( g(y_i) = y_{\sigma(i)} \) for every \( 1 \leq i \leq k \).

Using also that \( g(z) = z \) we get that

\[
z_{\sigma(i)} + b_{\sigma(i)} x_{\sigma(i)} + a_{\sigma(i)} y_{\sigma(i)} = g(z_i + b_i x_i + a_i y_i) = g(z_i) + b_i x_{\sigma(i)} + a_i y_{\sigma(i)},
\]

thus, \( g(z_i) = z_{\sigma(i)} + (b_{\sigma(i)} - b_i) x_{\sigma(i)} + (a_{\sigma(i)} - a_i) y_{\sigma(i)} \). Now, \( h = g_{\sigma(i)}^{-1} g h_i \in H_1 \) satisfies

\[
h(x_i) = x_i, \quad h(y_i) = y_i, \quad h(z_i) = z_i + (b_{\sigma(i)} - b_i) x_i + (a_{\sigma(i)} - a_i) y_i.
\]

By part (2/a) of Corollary 3.2 we conclude that \( b_{\sigma(i)} = b_i \) and \( a_{\sigma(i)} = a_i \) for every \( 1 \leq i \leq k \). In other words, \( \sigma \) fixes both \( (a_1, \ldots, a_k) \) and \( (b_1, \ldots, b_k) \). By the definition of these vectors we get that \( \sigma = 1 \), i.e. \( g \in \cap_i H_i = N \). Furthermore, for every \( 1 \leq i \leq k \) we also have

\[
g(x_i) = x_i, \quad g(y_i) = y_i, \quad g(z_i + b_i x_i + a_i y_i) = z_i + b_i x_i + a_i y_i.
\]

Since \( x_i, y_i, z_i + b_i x_i + a_i y_i \in V_i \) is a base for \( H_i/C_{H_i}(V_i) \), we get that \( g = \cap_i C_{H_i}(V_i) = 1 \).

Now we turn to the reduction of Theorem 1.2 to primitive groups. Notice that \( N \) is a \( p \)-solvable group and \( V \) is the sum of at least \( k \) irreducible \( F_q \)-modules, so we have \( |N| \leq 24^{-k/3}|V|^{d-1} \) by Section 4. By the last paragraph of Section 4, we may assume that \( q = 3 \) (and \( p = 3 \)). In particular, the permutation group \( G/N \leq S_k \) is 3-solvable, and so it does not contain any non-Abelian alternating composition factor. Now [18, Corollary 1.5] implies that \( |G/N| \leq 24^{k-1/3} \). But then \( |G| = |N||G/N| \leq 24^{-1/3}|V|^{d-1} \) which is exactly what we wanted.

6. Groups of semilinear transformations

In this section we reduce Theorems 1.1 and 1.2 to the case when every irreducible \( F_q \)-submodule of \( V \) is absolutely irreducible for any normal subgroup \( N \) of \( G \).

For this purpose let \( N < G \) be a normal subgroup of \( G \). Then \( V \) is a homogeneous \( F_q \)-module, so \( V = V_1 \oplus V_2 \oplus \cdots \oplus V_k \), where the \( V_i \)'s are isomorphic irreducible \( F_q \)-modules. Let \( T := \text{End}_{F_q}(V_1) \). Assuming that the \( V_i \)'s are not absolutely irreducible, \( T \) is a proper field extension of \( F_q \), and

\[
C_{\text{GL(V)}}(N) = \text{End}_{F_q}(V) \cap \text{GL(V)} \simeq \text{GL}(k, T),
\]

since \( \text{End}_{F_q}(V) \) is isomorphic to the matrix algebra \( M_k(T) \) by [7, Theorem 1.7.5] Furthermore, \( L = \mathbb{Z}(C_{\text{GL(V)}}(N)) \simeq \mathbb{Z}(\text{GL}(k, T)) \simeq T^k \). Now, by using \( L \), we can extend \( V \) to a \( T \)-vector space of dimension \( l := \dim_T V < \dim_{F_q} V \). As
$G \leq N_{GL(V)}(L)$, in this way we get an inclusion $G \leq GL(l, T)$. We proceed by proving the following theorem.

**Theorem 6.1.** For a proper field extension $T$ of $F_q$ let $G \leq GL(l, T)$ be a semilinear group acting on the $F_q$-space $V$ and let $H = G \cap GL(l, T)$. Suppose that $G$ is $p$-solvable and that $b(H) \leq t(|T|)$. Then $b(G) \leq t(|T|)$.

**Proof.** We modify the proof of [12, Lemma 6.1] to make it work in this more general setting.

Clearly we may assume that $|T| \geq 8$ is different from a prime. In these cases $t(|T|) = 2$.

Let $u_1, u_2$ be a base for $H$. By Corollary 3.2, we may also assume that

$$N_H(\langle u_1 \rangle) \cap N_H(\langle u_1, u_2 \rangle) \subseteq N_H(\langle u_2 \rangle)$$

or that every non-identity element of $C_H(u_1) \cap N_H(\langle u_1, u_2 \rangle)$ takes $u_2$ to $u_2 + u_1$.

(The latter case occurs only if $p = 2$.)

For every $\alpha \in T$ let $H_\alpha = C_G(u_1) \cap C_G(u_2 + \alpha u_1) \leq G$. Our goal is to prove that $H_\alpha = 1$ for some $\alpha \in T$. If $g \in \langle \cup H_\alpha \rangle$, then $g(u_1) = u_1$ and $g(u_2) = u_2 + \delta u_1$ for some $\delta \in T$.

We claim that $|\langle \cup H_\alpha \rangle \cap H| \leq 2$. Let $h \in \langle \cup H_\alpha \rangle \cap H$. On the one hand, the action of $h$ on $V$ is $T$-linear, since $h \in H$. On the other hand, $h(u_1) = u_1$ and $h(u_2) = u_2 + \delta u_1$ for some $\delta \in T$. By our assumption above, either $h \in N_H(\langle u_2 \rangle)$ and $\delta = 0$, or $h$ is an involution and $\delta = 1$. Thus we obtain the claim since $C_H(u_1) \cap N_H(\langle u_2 \rangle) = 1$.

Let $z$ be the generator of the group $\langle \cup H_\alpha \rangle \cap H$. This is a central element in $\langle \cup H_\alpha \rangle$. For every $g \in G$ let $\sigma_g \in \text{Gal}(T/F_q)$ denote the action of $g$ on $T$.

Let $g$ and $h$ be two elements of $\langle \cup H_\alpha \rangle$. Since $G/H$ is embedded into $\text{Gal}(T/F_q)$, we get $\sigma_g \neq \sigma_h$ unless $g = h$ or $g = h z$. Furthermore, a routine calculation shows that the subfields of $T$ fixed by $\sigma_g$ and $\sigma_h$ are the same if and only if $(g) = (h)$ or $(g) = (h z)$.

If $g \in H_\alpha \cap H_\beta$, then $g(u_2) = u_2 + (\alpha - \alpha^\sigma)u_1 = u_2 + (\beta - \beta^\sigma)u_1$, so $\alpha - \beta$ is fixed by $\sigma_g$. Let $K_g = \{ \alpha \in T \mid g \in H_\alpha \}$. The previous calculation shows that $K_g$ is an additive coset of the subfield fixed by $\sigma_g$, so $|K_g| = p^d$ for some $d \mid f = \log_q |T|$. Since for any $d \mid f$ there is a unique $p^d$-element subfield of $T$, we get $|K_g| \neq |K_h|$ unless the subfields fixed by $\sigma_g$ and $\sigma_h$ are the same. As we have seen, this means that $(g) = (h)$ or $(g) = (h z)$. Consequently, $|K_g| \neq |K_h|$ unless $K_g = K_h$ or $K_g = K_{h z}$. Hence we get

$$|\bigcup_{g \in \cup H_\alpha \setminus \{1\}} K_g| \leq 2 \sum_{d \mid f, d < f} q^d \leq 2 \sum_{d < f} q^d < q^f = |T|,$$

So there is a $\gamma \in T$ which is not contained in $K_g$ for any $g \in \cup H_\alpha \setminus \{1\}$. This exactly means that $H_\gamma = C_G(u_1) \cap C_G(u_2 + \gamma u_1) = 1$. \hfill \Box

Using Theorem 6.1, we can assume that $G \leq GL(l, T)$. As $l = \dim_T V < \dim_{F_q}(V)$, we can use induction on the dimension of $V$, thus $b(G) \leq 2$.

By the last paragraph of Section 4, we need not consider Theorem 1.2 here. Hence in the following we assume that $V$ is a direct sum of isomorphic absolutely irreducible $F_qN$-modules for any $N \prec G$. 

7. Stabilizers of tensor product decompositions

Let $N < G$ and let $V = V_1 \oplus \cdots \oplus V_k$ be a direct decomposition of $V$ into isomorphic absolutely irreducible $F_q N$-modules. By choosing a suitable basis in $V_1, V_2, \ldots, V_k$, we can assume that $G \leq GL(n, q)$ such that any element of $N$ is of the form $A \otimes I_k$ for some $A \in N_{V_1} \leq GL(n/k, q)$. By using [14, Lemma 4.4.3(ii)] we get

$$N_{GL(n,q)}(N) = \{B \otimes C \mid B \in N_{GL(n/k,q)}(N_{V_1}), C \in GL(k,q)\}.$$

Let

$$G_1 = \{g_1 \in GL(n/k, q) \mid \exists g \in G, g_2 \in GL(k,q) \text{ such that } g = g_1 \otimes g_2\}.$$

We define $G_2 \leq GL(k,q)$ in an analogous way. Then $G \leq G_1 \otimes G_2$. Here $G/Z \simeq (G_1/Z) \times (G_2/Z)$, hence $G_1 \leq GL(n/k,q)$ and $G_2 \leq GL(k,q)$ are $p$-solvable irreducible linear groups. If $1 < k < n$, then by using induction for $G_1 \leq GL(n/k,q)$ and $G_2 \leq GL(k,q)$ we get $b(G_1) \leq t(q)$ and $b(G_2) \leq t(q)$. Furthermore $b^*(G_1) \leq t(q)$ and $b^*(G_2) \leq t(q)$ by Corollary 3.3. Thus [16, Lemma 3.3 (ii)] gives us

$$b(G) \leq b(G_1 \otimes G_2) \leq b^*(G_1 \otimes G_2) \leq \max(b^*(G_1), b^*(G_2)) \leq t(q).$$

For the reduction of Theorem 1.2, by using induction on the dimension, we have

$$|G| \leq |G_1| \cdot |G_2| \leq 24^{-1/3}q^{(n/k)(d-1)} \cdot 24^{-1/3}q^{k(d-1)} \leq 24^{-1/3}|V|^{d-1}.$$

Thus, from now on we can assume that for every normal subgroup $N < G$ either $N \leq Z$ or $V$ is absolutely irreducible as an $F_q N$-module.

8. Groups of symplectic type

From now on assume that $N$ is a normal subgroup of $G$ containing $Z$ such that $N/Z$ is a minimal normal subgroup of $G/Z$. Then $N/Z$ is a direct product of isomorphic simple groups. In this section we examine the situation when $N/Z$ is an elementary Abelian group.

If $N$ is Abelian then it is central in $G$. So assume that $N$ is non-Abelian.

If $N/Z$ is elementary Abelian of rank at least 2, then $G$ is of symplectic type. Such groups were examined in [12, Section 5] (see also Remark 5.20 in [12]) where it was proved that $b(G) \leq 2$ unless $q \in \{3, 4\}$, when $b(G) \leq 3$ holds.

For the reduction of Theorem 1.2, we need only examine the case $q = 3, n = 2^k$. For this we can use the fact that $G/N$ can be considered as a subgroup of the symplectic group $Sp(2k, 2)$. By the theorem of Pálfy [19] and Wolf [23], we may assume that $G$ is a non-solvable (and 3-solvable) group. Thus we must have a composition factor of $G$ (and thus of $G/N$) isomorphic to a Suzuki group. Since the smallest Suzuki group $Suz(8)$ has order larger than $|Sp(4, 2)|$, we must have $k \geq 3$. On the other hand, since the second largest Suzuki group $Suz(32)$ has order larger than $|Sp(6, 2)|$ and since $Suz(8)$ is not a section of $Sp(6, 2)$ (since 13 divides the order of the first group but not the order of the second), we see that $k \neq 3$.

But for $k \geq 4$ we clearly have $|G| = |N||G/N| < 2^{2k^2+3k+3} < 24^{-1/3}|V|^{d-1}$, by use of the formula for the order of $Sp(2k, 2)$. 


9. Tensor product actions

Now let \( N/Z \) be a direct product of \( t \geq 2 \) isomorphic non-Abelian simple groups. Then \( N = L_1 \rtimes L_2 \rtimes \cdots \rtimes L_t \) is a central product of isomorphic groups such that for every \( 1 \leq i \leq t \) we have \( Z \leq L_i \), \( L_i/Z \) is simple. Furthermore, conjugation by elements of \( G \) permutes the subgroups \( L_1, L_2, \ldots, L_t \) in a transitive way. By choosing an irreducible \( \mathbb{F}_q L_1 \)-module \( V_1 \leq V \), and a set of coset representatives \( g_1 = 1, g_2, \ldots, g_t \in G \) such that \( L_i = g_i L_1 g_i^{-1} \), we get that \( V_1 := g_i V_1 \) is an absolutely irreducible \( \mathbb{F}_q L_1 \)-module for each \( 1 \leq i \leq t \). Now, \( V \simeq V_1 \otimes V_2 \otimes \cdots \otimes V_t \) and \( G \) permutes the factors of this tensor product. It follows that \( G \) is embedded into the central wreath product \( G_1 \wr S_t \) defined to be a split extension of the base group \( G_1 \otimes G_1 \otimes \cdots \otimes G_1 \) by \( S_t \). Clearly \( G_1 \leq GL(V_1) \)

is a \( p \)-solvable irreducible linear group. Thus \( b(G_1) \leq t(q) \) and \( b^*(G_1) \leq t(q) \) by induction on the dimension \( m \) of \( V_1 \) and by Corollary 3.3.

First let \( q \geq 5 \). Then \( t(q) = 2 \). Thus \( b(G) \leq 2 \) follows from [12, Theorem 3.6] unless \( (m,t) = (2,2) \). In case \( (m,t) = (2,2) \), that is, \( G \leq G_1 \wr S_2 \leq GL(4,q) \) for some \( p \)-solvable group \( G_1 \leq GL(2,q) \) let \( x_1, y_1 \in V_1 \) be a basis of \( V_1 \) satisfying either \( N_{G_1}((x_1)) \subseteq N_{G_1}((y_1)) \) or the property that every non-identity element of \( C_{G_1}(x_1) \) takes \( y_1 \) to \( y_1 + x_1 \). (Such a basis exists by Theorem 3.1.) We claim that if \( a \in \mathbb{F}_q \setminus \{0,1\} \) then \( x_1 \otimes x_1, y_1 \otimes (y_1 + \alpha x_1) \) is a base for \( G_1 \wr S_2 \geq G \). Indeed, let \( g = (A \otimes B) \sigma \in G_1 \wr S_2 \) with \( A, B \in G_1 \), \( \sigma \in S_2 \) fixing these two vectors. Then \( g(x_1 \otimes x_1) = x_1 \otimes x_1 \) implies that \( Ax_1 = \lambda x_1, Bx_1 = \lambda^{-1}x_1 \) for some \( \lambda \in \mathbb{F}_q^\times \). If \( N_{G_1}((x_1)) \subseteq N_{G_1}((y_1)), \) then \( Ay_1 = ay_1, By_1 = by_1 \) for some \( a, b \in \mathbb{F}_q^\times \). Hence

\[
y_1 \otimes (y_1 + \alpha x_1) = g(y_1 \otimes (y_1 + \alpha x_1)) = aby_1 \otimes y_1 + \alpha a \lambda^{-1} (y_1 \otimes x_1)^\sigma.
\]

Comparing the coefficients of \( y_1 \otimes y_1 \) and \( y_1 \otimes x_1 \) in the above equality we get \( ab = a \lambda^{-1} = 1 \) and \( \sigma = 1 \). So, \( A = \lambda I, \ B = \lambda^{-1} I \) and \( g = 1 \), as claimed. Similarly, if every non-identity element of \( C_{G_1}(x_1) \) takes \( y_1 \) to \( y_1 + x_1 \), then by multiplying \( A \) with \( \lambda^{-1} \) and \( B \) with \( \lambda \), we can assume that \( \lambda = 1 \). Then for some \( \varepsilon_a, \varepsilon_b \in \{0,1\} \) we have

\[
y_1 \otimes (y_1 + \alpha x_1) = g(y_1 \otimes (y_1 + \alpha x_1)) = \left( (y_1 + \varepsilon_a x_1) \otimes (y_1 + (\alpha + \varepsilon_b) x_1) \right)^\sigma.
\]

Comparing the coefficients of \( x_1 \otimes x_1, x_1 \otimes y_1 \) and \( y_1 \otimes x_1 \) we get \( \varepsilon_a = \varepsilon_b = 0, \ \sigma = 1, \) so \( g = 1 \) follows.

Now, let \( q = 3 \). Let \( x_1, y_1, z_1 \in V_1 \) be a strong base for \( G_1 \). Then the stabilizer of \( x_1 \otimes x_1 \otimes \cdots \otimes x_1 \in V \) is of the form \( H = H_1 \wr S_t \), where \( y_1, z_1 \in V_1 \) is a strong base for \( H_1 = N_{G_1}(x_1) \), so \( b^*(H_1) \leq 2 \). If \( (m,t) \neq (2,2) \) then \( b(H) \leq 2 \) by [12, Theorem 3.6], which results in \( b(G) \leq 3 \). Finally, let \( (m,t) = (2,2) \). By choosing a basis \( x_1, y_1 \in V_1 \), it is easy to see that \( x_1 \otimes x_1, y_1 \otimes y_1, x_1 \otimes y_1 \in V \) is a base for \( GL(V_1) \wr S_2 \geq G \).

As for the order of \( G \) notice that \( G \leq G_1 \wr S_t \) where \( S \leq S_t \) is a \( 3 \)-solvable group. Thus by induction and by [18, Corollary 1.5] we have

\[
|G| \leq |G_1|^t |S| \leq 24^{-t/3} |V_1|^{(d-1)/t} 24^{(t-1)/3} = 24^{-1/3} |V|^{d-1}.
\]
10. Almost quasisimple groups

Finally, let $Z \leq N < G$ be such that $N/Z$ is a non-Abelian simple group. Let $N_1 = [N,N] \triangleleft G$ and let $V_1$ be an irreducible $\mathbb{F}_p N_1$-submodule of $V$ and $G_1 = \{ g \in G \mid g(V_1) = V_1 \}$ be the stabilizer of $V_1$. By using the same argument as in the last paragraph of [12, Page 29] we get that $G_1$ is included in $GL(V_1)$ and we have a chain of subgroups $N_1 \vartriangleleft G_1 \leq GL(V_1)$ where $G_1$ is $p$-solvable, $N_1$ is quasisimple and $V_1$ is irreducible as an $\mathbb{F}_p N_1$-module.

Suppose that $b(G_1) \leq 2$ in the action of $G_1$ on $V_1$, that is, there exist $x,y \in V_1 \leq V$ such that $C_{G_1}(x) \cap C_{G_1}(y) = 1$. For any element $g \in G$ with $g(x) = x$ we have that $N_1 x = \{ n x \mid n \in N_1 \}$ is a $g$-invariant subset. As the $\mathbb{F}_p$-subspace generated by $N_1 x$ is exactly $V_1$, we get that $g \in G_1$. This proves that $C_{G}(x) \cap C_{G}(y) = C_{G_1}(x) \cap C_{G_1}(y) = 1$. Thus $b(G) \leq 2$.

Hence if we manage to show that $b(G_1) \leq 2$ then we are finished with the proofs of both Theorems 1.1 and 1.2.

So assume that $G = G_1$, $N = N_1$, and $V = V_1$. By the first three paragraphs of this section, we have that $N \leq p$. To summarize, $G \leq GL(V)$ is a group having a quasisimple irreducible normal subgroup $N$ and $Z \leq G$.

We can assume that $G/Z$ is almost simple. For this it is sufficient to see that $N/Z$ is the unique minimal normal subgroup of $G/Z$. For let $M/Z$ be another minimal normal subgroup of $G/Z$. By Section 8, we may assume that $M/Z$ is non-Abelian. Furthermore the group $MN$ is a central product and so $[M,N] = 1$. But this is impossible since the centralizer of $N$ in $G$ must be Abelian.

Lemma 10.1. If $N$ has a regular orbit on $V$ then $b(G) \leq 2$.

Proof. Since $N$ is normal in $G$, a regular $N$-orbit $\Delta$ containing a given vector $v$ is a block of imprimitivity inside the $G$-orbit containing $v$. Hence the group $C_{G}(v)N$ is transitive on $\Delta$ and $N$ is regular on $\Delta$. Thus for every $h \in C_{G}(v)$ the number $|fix(h)|$ of fixed points of $h$ on $\Delta$ is $|C_N(h)|$. To prove that $G$ has a base at size $N$ on $V$, it is sufficient to see that there exists a vector $w$ in $\Delta$ that is not fixed by any non-trivial element of $C_G(v)$.

First notice that if $N/Z(N)$ is isomorphic to the non-Abelian finite simple group $S$ then $|C_G(v)| \leq |\text{Out}(S)| < m(S)$ where $m(S)$ is the minimal index of a proper subgroup of $S$. This latter inequality follows from [1, Lemma 2.7 (i)].

But

$$\sum |\text{fix}(h)| = \sum |C_N(h)| < |C_G(v)| \frac{|N|}{m(S)} < |N|$$

where the sums are over all non-identity elements $h$ in $C_G(v)$. This completes the proof of the lemma.

By Lemma 10.1, in the following we may assume that $N$ does not have a regular orbit on $V$. Our final theorem finishes the proofs of Theorems 1.1 and 1.2.

Theorem 10.2. Under the current assumptions $G$ is a $p'$-group and $b(G) \leq 2$.

Proof. By using Goodwin’s theorem [11, Theorem 1], Köhler and Pahlings [15, Theorem 2.2] gave a complete list of (irreducible) quasisimple $p'$-groups $N$ such that $N$ does not have a regular orbit on $V$. In all these exceptional cases, when $N/Z$ is simple, $|\text{Out}(N/Z)|$ is divisible by no prime larger than $3$ while $p$ is always at least $5$. So $G$ itself is a $p'$-group. But then $G$ admits a base of size $2$ on $V$ by [12, Theorem 4.4].
THE MINIMAL BASE SIZE

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