Lie symmetry analysis of the equation $\ddot{u} = f_0(u) + f_1(u)\dot{u}$

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- FritzHugh-Nagumo model:

$$\dot{v} = v(1-v)(v-a) - w + I, \tag{1}$$
$$\dot{w} = bv - \gamma w, \tag{2}$$

- Expressing w from (1) and substituting into (2) we have

$$\ddot{u} = f_0(u) + f_1(u)\dot{u}, \tag{3}$$

where:

$$f_0(u) = \gamma(-u^3 + u^2 - ua + u^2 a) - bu$$
$$f_1(u) = -3u^2 + 2u(1+a) - a - \gamma$$

- We seek the infinitesimal generator of a symmetry in the form

$$X = \xi(t,u)\frac{\partial}{\partial t} + \eta(t,u)\frac{\partial}{\partial u}.$$

- The symmetry condition:

$$[X^1 - \xi S^1, S^1] = 0,$$

where

$$X^1 = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + (D\eta - \dot{u}D\xi)\frac{\partial}{\partial \dot{u}}.$$

$$S^1 = \frac{\partial}{\partial t} + \dot{u}\frac{\partial}{\partial u} + (f_0(u) + f_1(u)\dot{u})\frac{\partial}{\partial \dot{u}},$$

- The symmetry condition gives that

$$\begin{aligned}&\left(f_0'\eta + f_1\eta_t - \eta_{tt} + 2\xi_t f_0 - f_0\eta_u\right)\\&+ \left(f_1'\eta + f_1\xi_t - 2\eta_{tu} + \xi_{tt} + 3f_0\xi_u\right) \cdot \dot{u}\\&+ \left(-\eta_{uu} + 2\xi_{tu} + 2f_1\xi_u\right) \cdot (\dot{u})^2\\&+ \xi_{uu} \cdot (\dot{u})^3 = 0.\end{aligned}$$

- Hence we obtain

$$f_0'(u)\eta + f_1(u)\eta_t - \eta_{tt} + 2\xi_t f_0(u) - f_0(u)\eta_u = 0, \qquad (4)$$

$$f_1'(u)\eta + f_1(u)\xi_t - 2\eta_{tu} + \xi_{tt} + 3f_0(u)\xi_u = 0, \qquad (5)$$

$$-\eta_{uu} + 2\xi_{tu} + 2f_1(u)\xi_u = 0, \qquad (6)$$

$$\xi_{uu} = 0. \qquad (7)$$

- We restrict our attention to the case $\xi_u(t,u) = 0$, $\eta_t(t,u) = 0$.
- $\xi_u(t,u) = 0 \Rightarrow \xi(t,u) = \xi(t)$, $\eta_t(t,u) = 0 \Rightarrow \eta(t,u) = \eta(u)$.
- From (6) we conclude that $\eta_{uu}(u) = 0$. So we obtain

$$\eta(u) = au + b.$$

- If $f_0(u) \neq 0$, then differentiating (4) with respect to t we have

$$2\xi_{tt}(t) f_0(u) = 0 \Rightarrow \xi = ct + d.$$

- So (4) gives that

$$f_0'(u)(au+b) = f_0(u)(a-2c) \Rightarrow f_0(u) = A_1(au+b)^{-\frac{2c}{a}+1},$$

- and from (5) we find that

$$f_1'(u)\eta(u) + f_1(u)c = 0 \Rightarrow f_1(u) = B_1(au+b)^{-\frac{c}{a}}. \qquad (8)$$

- If $f_1(u) = 0$, then (3) can be solved by elementary methods.

Theorem

Let $\mathcal{L}$ be the Lie subalgebra of all symmetries of $\ddot{u} = f_0(u) + f_1(u)\dot{u}$ defined by the equations $\xi_u = 0$, $\eta_t = 0$. If $f_0(u) = A_1(au+b)^{-\frac{2c}{a}+1}$, $A_1 \neq 0$ and $f_1(u) = B_1(au+b)^{-\frac{c}{a}}$ then $\mathcal{L}$ is two dimensional and the generators of $\mathcal{L}$ are $Y_1 = \frac{\partial}{\partial t}$ and $Y_2 = t\frac{\partial}{\partial t} - \frac{f_1(u)}{f_1'(u)}\frac{\partial}{\partial u}$.

If $f_0(u) = 0$, then differentiating (5) with respect to u we obtain

$$f_1''(u)\eta + f_1'(u)\eta_u + f_1'(u)\xi_t = 0 \tag{9}$$

Differentiating (9) with respect to t we have

$$f_1'(u)\xi_{tt} = 0. \tag{10}$$

Therefore $f_1'(u) = 0$ or $\xi_{tt} = 0$. If $f_1'(u) = 0$ then (3) takes the form $\ddot{u} = K\dot{u}$. If $\xi_{tt} = 0$ then $\xi = ct + d$ and we have that $f_1(u) = B_1(au + b)^{-\frac{c}{a}}$

Let us suppose that $f_0(u) = u^{1+2c}$, $f_1(u) = u^c$. Then (3) takes the form $u^{1+2c} + u^c\dot{u}$. An equivalent first order system is

$$\dot{u} = uv$$
$$\dot{v} = u^{2c} + u^c v - v^2.$$