Invariant metrizability and projective metrizability on Lie groups

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Canonical connections on Lie groups

- G Lie group
- Canonical connections: ∇^-, ∇^+, ∇^0
- If X, Y are left-invariant vector fields:

  1. $\nabla^-_X Y = 0$,
  2. $\nabla^+_X Y = [X, Y]$,
  3. $\nabla^0_X Y = \frac{1}{2}[X, Y]$,

- Curvature, torsion:

  1. $R^- = 0$, $T^- \neq 0$,
  2. $R^+ = 0$, $T^+ \neq 0$,
  3. $T^0 = 0$, $R^0(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$,

- **Problem**: Find a left invariant lagrangian $F : TG \to \mathbb{R}$ which is l-homogeneous and satisfies the Euler-Lagrange equation associated to the canonical connection of G.

The geometric structure

- $\nabla \;\Rightarrow\; v \;\Rightarrow\; h \;\Rightarrow\; S \;\Rightarrow\; \ddot{x}^i = f^i(x,\dot{x}) \;\Rightarrow\; \ddot{x} = \dot{x}x^{-1}\dot{x}$

- Invariant coordinates:

  · $L_x{:}G \to G, \;\Rightarrow\; L_{x^{-1}}{:}T_xG \to \mathfrak{g} \;\Rightarrow\; T_xG \simeq \mathfrak{g} \;\Rightarrow\; TG \simeq G \times \mathfrak{g}$

  · Coordinates on TG: $(x,y) \;\Rightarrow\; (x,\alpha) \qquad (\alpha := L_{x^{-1}}dx)$

  Left translation: $L_g(x,y) = (gx, gy) \;\Leftrightarrow\; L_g(x,\alpha) = (gx,\alpha)$

  · Coordinates on TTG: $(x,\alpha) \to (x,\alpha,X,A)$:

$$(x,\alpha,X,A) \simeq X\frac{\partial}{\partial x}\bigg|_{(x,\alpha)} + A\frac{\partial}{\partial \alpha}\bigg|_{(x,\alpha)} \simeq X^i\frac{\partial}{\partial x^i}\bigg|_{(x,\alpha)} + A^i\frac{\partial}{\partial \alpha^i}\bigg|_{(x,\alpha)}$$

  · Left translation on TTG: $L_g(x,\alpha,X,A) = (gx,\alpha,gX,A)$

Using "invariant" coordinate system (x, α) and (x, α, X, A):

- Left invariant vector field: $\tilde{a} : G \to TG, \quad x \to (x, a)$.
- $V_{(x,\alpha)}TG = \big\{(x, \alpha, 0, b) \,|\, b \in \mathfrak{g}\big\}$
- $v(x, \alpha, xa, c) = (x, \alpha, 0, \frac{1}{2}[a, \alpha] + c)$,
- $h(x, \alpha, xa, c) = (x, \alpha, xa, -\frac{1}{2}[a, \alpha])$
- $\tilde{a}^h\big|_{(x,\alpha)} = (x, \alpha, xa, -\frac{1}{2}[a, \alpha]) = xa\dfrac{\partial}{\partial x} - \frac{1}{2}[a, \alpha]\dfrac{\partial}{\partial \alpha}$
- $\tilde{a}^v\big|_{(x,\alpha)} = (x, \alpha, 0, a) = a\dfrac{\partial}{\partial \alpha}$
- $S(x, \alpha) = (x, \alpha, x\alpha, 0) = x\alpha\dfrac{\partial}{\partial x}$

Invariant l-homogeneous Euler-Lagrange equation on Lie groups

Problem: Find a left invariant lagrangian $F : TG \to \mathbb{R}$ which is ℓ-homogeneous and satisfies the Euler-Lagrange equation associated to the canonical spray of G.

- $\omega_F := i_S \Omega_F + d\mathcal{L}_C F - dF$
- $\omega_F \equiv 0$
- $\omega_F(\tilde{a}^h) = [a, \alpha] \dfrac{\partial F}{\partial \alpha} + x\alpha a \dfrac{\partial^2 F}{\partial x \partial \alpha} - xa \dfrac{\partial F}{\partial x}$

PDE system on $\mathbf{F} : TG \to \mathbb{R}$

$$\frac{\partial F}{\partial x} = 0, \qquad \alpha \frac{\partial F}{\partial \alpha} - \ell \cdot F = 0, \qquad [a, \alpha] \frac{\partial F}{\partial \alpha} = 0, \quad \forall a \in \mathfrak{g},$$

PDE system on $\mathcal{F} : \mathfrak{g} \to \mathbb{R}$

$$\alpha \frac{\partial \mathcal{F}}{\partial \alpha} - \ell \cdot \mathcal{F} = 0, \qquad [a, \alpha] \frac{\partial \mathcal{F}}{\partial \alpha} = 0, \quad \forall a \in \mathfrak{g},$$

On the integrability of the canonical left invariant l-homogeneous Euler-Lagrange system

- $\Delta_{\mathfrak{g}} := \left\langle X_a(\alpha) = [a, \alpha] \frac{\partial}{\partial \alpha} \;\middle|\; a \in \mathfrak{g} \right\rangle \qquad \subset T\mathfrak{g}$
- $\pi : T\mathfrak{g} \to T\mathfrak{g}$ projection on $\Delta_{\mathfrak{g}}$

System to investigate:

$$P_1 \mathcal{F} := \mathcal{L}_C \mathcal{F} - \ell \mathcal{F} \equiv 0, \qquad\qquad P_2 \mathcal{F} := d_\pi \mathcal{F} \equiv 0,$$

Properties:

- $[X_a, X_b] = X_{[a,b]} \quad \Rightarrow \quad \Delta_{\mathfrak{g}}$ is involutive
- P_1 and P_2 are compatible $\quad \Rightarrow \quad P = (P_1, P_2)$ is integrable.

To prove the integrability of P: calculation of the symbols, prolongations, obstructions, find a quasi-regular basis – involutivity of the symbols.

About the metrizability and projective metrizability of the canonical connection

Theorem: Let $\alpha \in \mathfrak{g}$ a generic element of $\mathfrak{g}$ and consider the system of linear equations:

$$\left.\begin{aligned} \alpha_k x_k - \ell x &= 0, & & \\ C^k_{ij}\alpha_j x_k &= 0, & i &= 1, ..., n, \\ \alpha_k x_{ki} &= 0, & i &= 1, ..., n, \\ C^k_{ij} x_k + C^k_{jm}\alpha_m x_{ik} &= 0, & i, j &= 1, ..., n, \end{aligned}\right\} \tag{1}$$

1. The canonical connection of G is locally Finsler metrizable with a left invariant Finsler function iff the linear system (1) with $\ell = 2$ has a non-degenerate solution $\{x = \epsilon, x_i = \epsilon_i, x_{ij} = \epsilon_{ij}\}$. /Non-degenerate $\Leftrightarrow$ (ϵ_{ij}) is positive definite./
2. The canonical connection of G is locally projectively equivalent to a left-invariant Finsler metric iff the linear system (1) with $\ell = 1$ has a solution $\{x = \epsilon, x_i = \epsilon_i, x_{ij} = \epsilon_{ij}\}$, where (ϵ_{ij}) positive semi-definite, $\alpha \in \ker(\epsilon_{ij})$, $\mathrm{rank}(\epsilon_{ij}) = n - 1$.

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