About the holonomy and curvature of Finsler manifolds

joint work with P.T. Nagy

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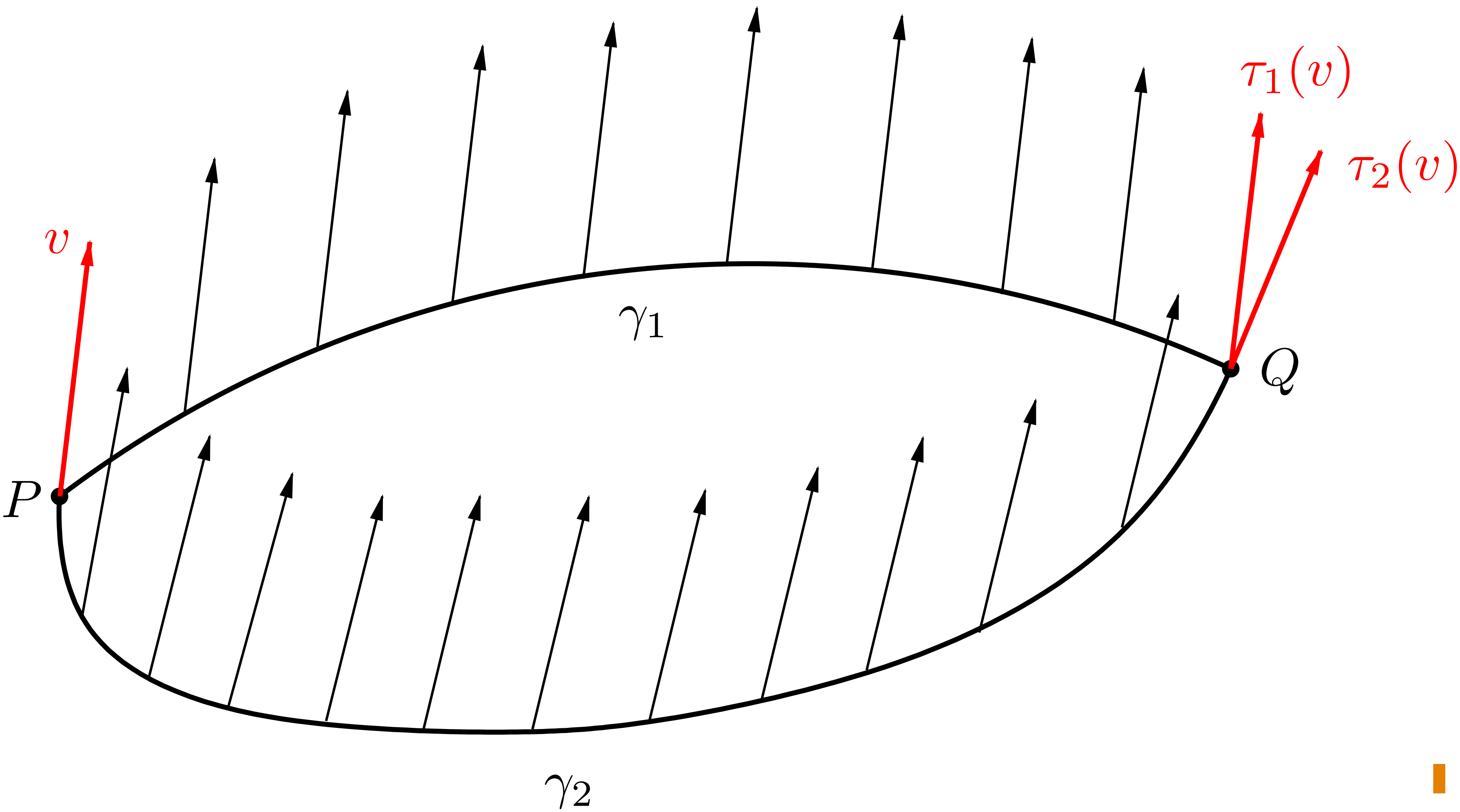
SEVENTH FRAMEWORK PROGRAMME

Abstract

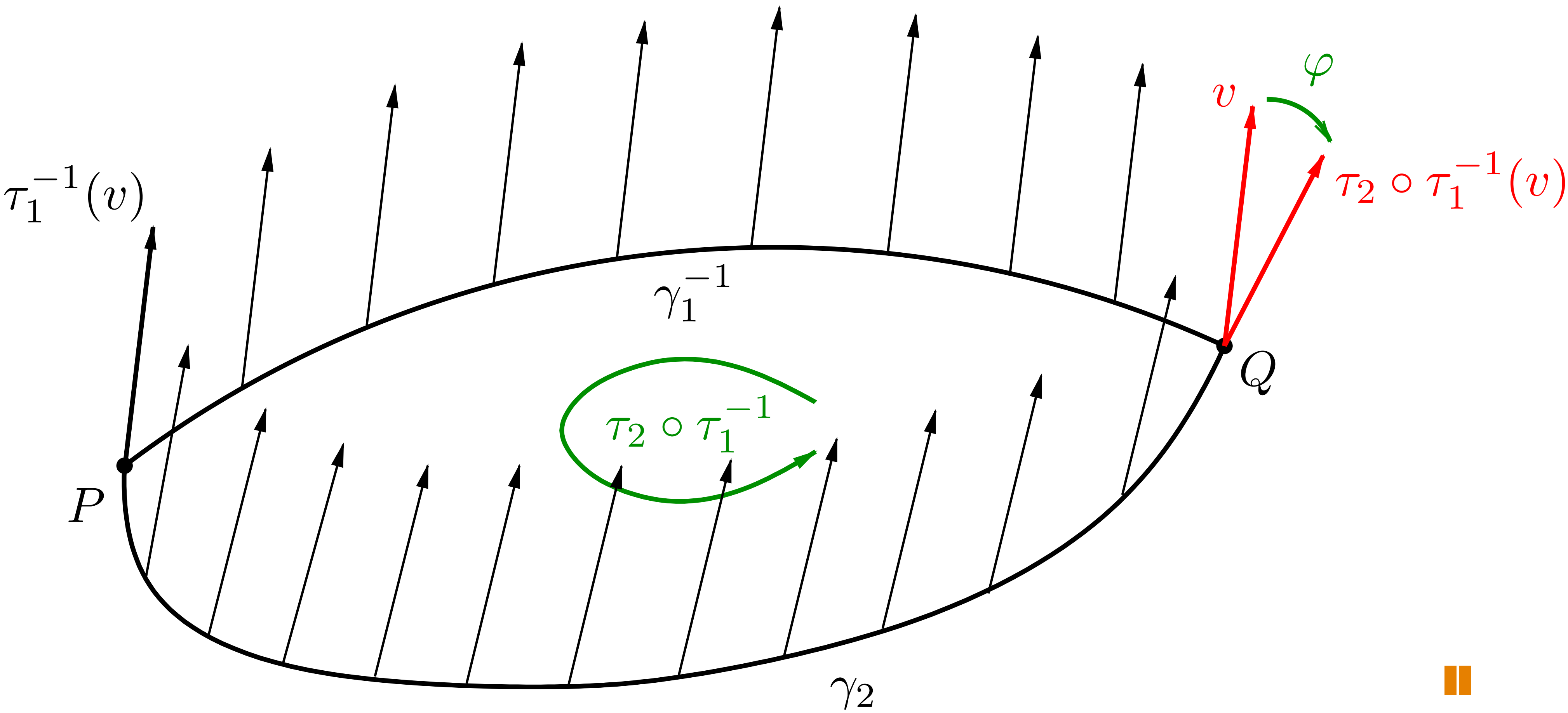
The aim of this talk is to present some new results about the holonomy of the Finsler manifolds

- Finsler manifolds,
- Holonomy group,
- Tangent Lie algebras to the holonomy group,
- The **curvature algebra** and **infinitesimal holonomy algebra**,
- New results about the holonomy of the Finsler manifolds.

Parallel translation along curves

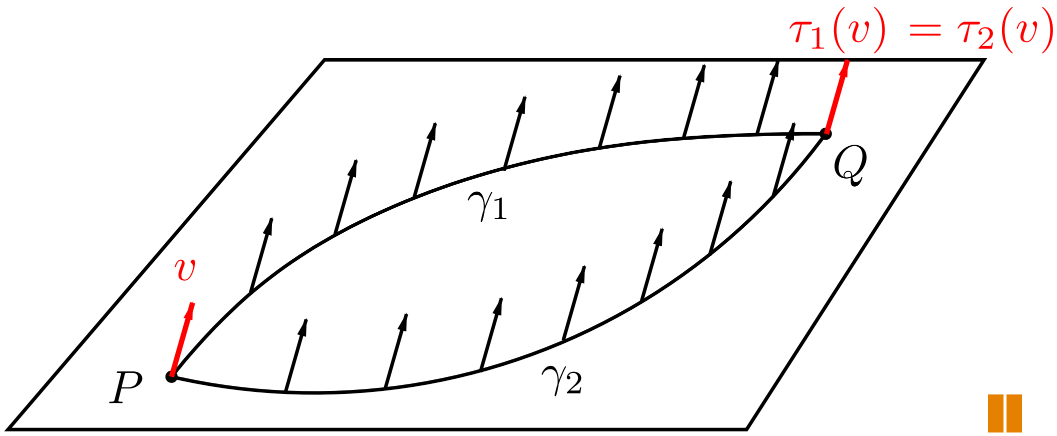


Holonomy


$\tau_1^{-1}(v)$
v
φ
$\tau_2 \circ \tau_1^{-1}(v)$
γ_1^{-1}
$\tau_2 \circ \tau_1^{-1}$
P
Q
γ_2

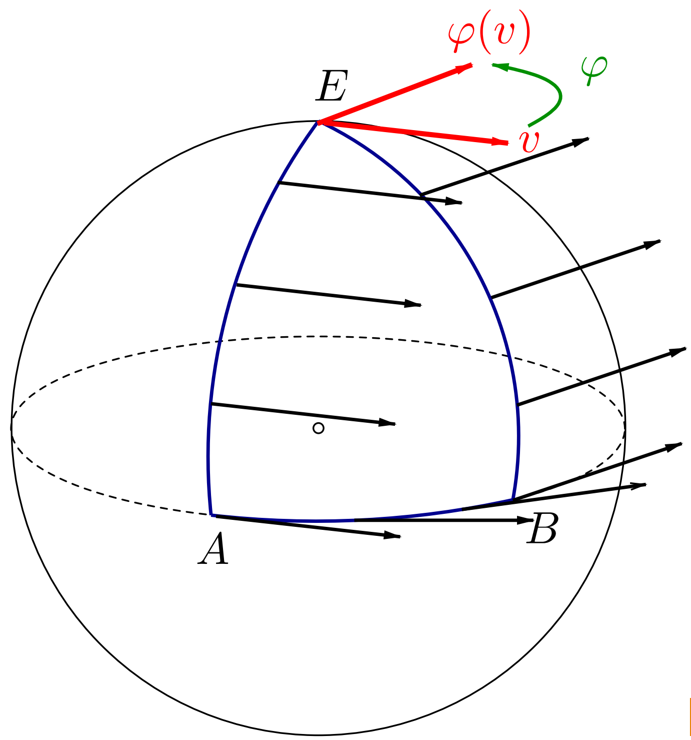
Examples

1. Plane:



$$\mathbf{R} \equiv \mathbf{0}$$

2. Sphere:



$$\mathbf{R} \not\equiv \mathbf{0}$$

Parallel translation, holonomy

- Riemannian manifolds: $\langle\ ,\ \rangle_{\mathbf{p}} = g_{ij}(\mathbf{x})dx^i \otimes dx^j$
- Finslerian manifolds: $\langle\ ,\ \rangle_{(\mathbf{p},\mathbf{v})} = g_{ij}(\mathbf{x},\mathbf{y})dx^i \otimes dx^j$
- Geodesics: $\ddot{x}^i + 2G^i(x,\dot{x}) = 0,$

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^j y^k.$$

- Parallel vector field $X(t)$ along a curve $c(t)$:

$$\nabla_{\dot{c}}X(t) = \Big(\frac{dX^i(t)}{dt} + \Gamma^i_j(c(t),X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \quad \Gamma^i_j = \frac{\partial G^i}{\partial y^j}.$$

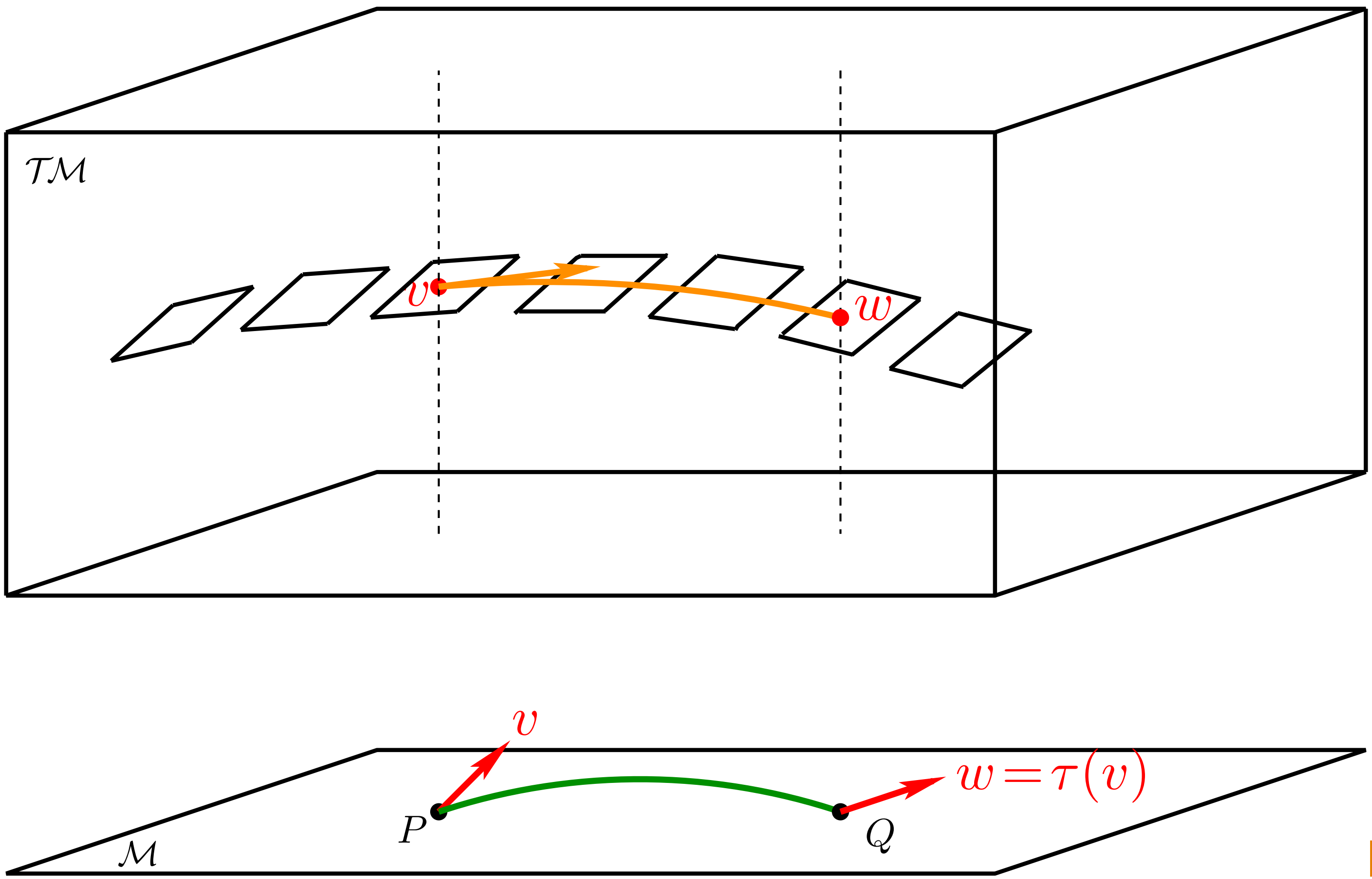
- Parallel translation along a curve $c\colon[0,1]\to M$:

$\tau_c\colon T_{c_0}M \to T_{c_1}M,\qquad X_0 \Rightarrow X(t) \Rightarrow \tau X_0 = X(1) \Rightarrow \tau_c : \mathcal{I}_{c_0} \to \mathcal{I}_{c_1}$

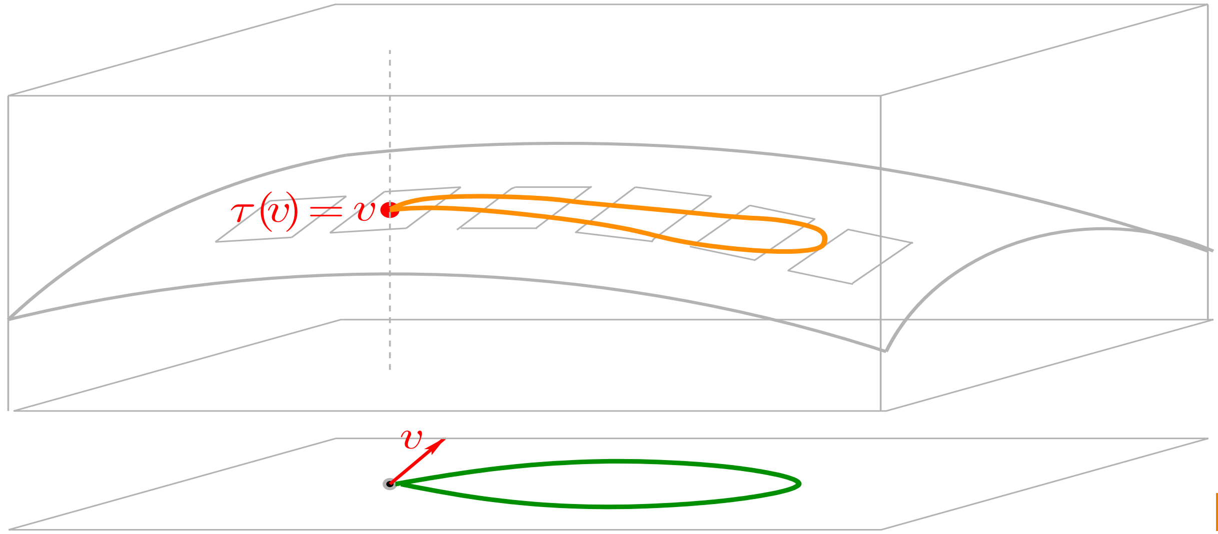
- The *holonomy group* is generated by parallel translation along closed curves

$\Rightarrow$ subgroup of Diff$^\infty(\mathcal{I}_x)$ determined by parallel translations.

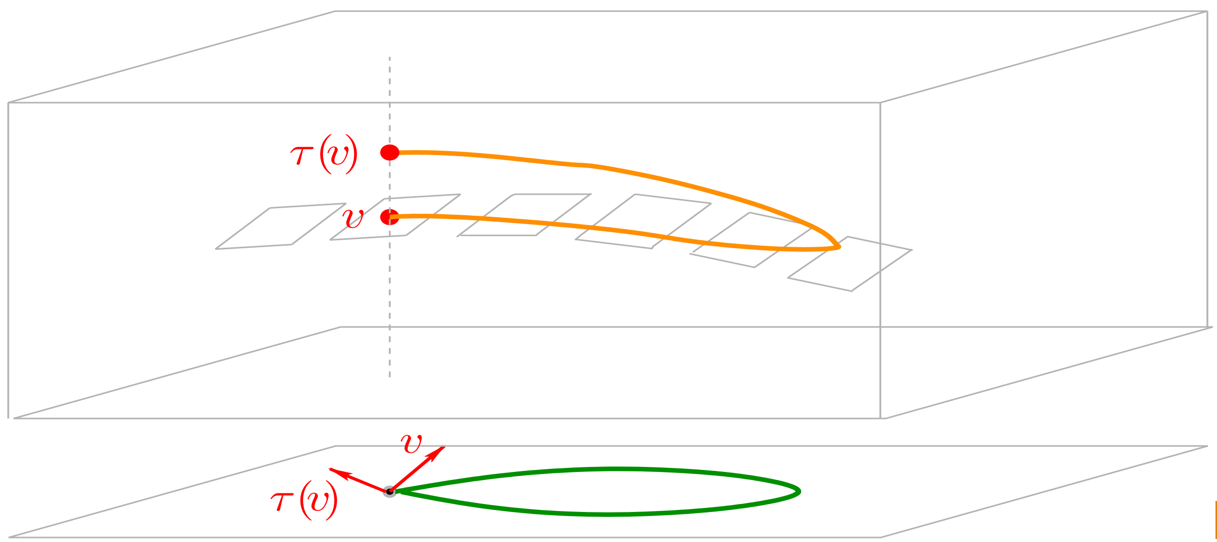
Parallel translation: geometric construction



- $R \equiv 0$



- $R \not\equiv 0$



Tangent Lie algebras to a subgroup H of $\mathrm{Diff}^\infty(\mathcal{I})$

Def: • A vector field X is called ***tangent*** to $H \subset \mathrm{Diff}^\infty(\mathcal{I})$, if there exists $\{\phi_t\}$ in H such that

$$\phi_0 = \mathrm{Id}, \qquad \left.\frac{\partial \phi_t}{\partial t}\right|_{t=0} = X.$$

• A Lie algebra $\mathfrak{h}$ is ***tangent*** to H, if its elements are tangent to H.

$\mathfrak{h}$ tangent to $H \quad \Rightarrow \quad$ information on H

Prop: If $\mathfrak{h}$ is tangent to a closed subgroup H of $\mathrm{Diff}^\infty(\mathcal{I})$, then $\exp(\mathfrak{h}) \subset H$.

Proof: • $X \in \mathfrak{h}$, the 1-parameter family of diffeomorphisms $\{\phi(t) \in H\}_{t\in\mathbb{R}}$:

$$\left\{ \left(\phi\left(\tfrac{t}{n}\right)\right)^n \in H \right\}_{t\in\mathbb{R}} \longrightarrow \{\exp tX\}_{t\in\mathbb{R}}$$

• H is closed $\Rightarrow \{\exp tX\}_{t\in\mathbb{R}} \subset H$.

Definition: A vector field X is ***strongly tangent*** to a subgroup H, if there exists a $k \in \mathbb{N}$ and a smooth k-parameter family $\{\phi_{(t_1,\ldots,t_k)}\}$ of diffeomorphisms in H such that

1. $\phi_{(t_1,\ldots,t_k)} = \mathrm{Id}$, if $t_j = 0$ for some $1 \leq j \leq k$;
2. $\left.\dfrac{\partial^k \phi_{(t_1,\ldots,t_k)}}{\partial t_1 \ldots \partial t_k}\right|_{(t_1,\ldots,t_k)=(0,\ldots,0)} = X.$

Remark: If X is strongly tangent to H, then it is also tangent to H.

Proposition: *The Lie algebra generated by strongly tangent vector fields is tangent to H.*

$$\begin{array}{l} X_1 \text{ strongly tangent} \Rightarrow \{\phi^1_{(t_1,\ldots,t_{k_1})}\} \\ X_2 \text{ strongly tangent} \Rightarrow \{\phi^2_{(s_1,\ldots,s_{k_2})}\} \end{array} \Rightarrow \quad \left[\phi^1_{(t_1,\ldots,t_{k_1})}, \phi^2_{(s_1,\ldots,s_{k_2})}\right]$$

$$\left[\phi^1_{(t_1,\ldots,t_{k_1})}, \phi^2_{(s_1,\ldots,s_{k_2})}\right] \quad \Rightarrow \quad [X_1, X_2]$$

Curvature algebra and infinitesimal holonomy algebra

- $\mathfrak{R}(M)$: the *curvature algebra* is the smallest Lie algebra generated by curvature vector fields.

- $\mathfrak{hol}^*(M)$: the *infinitesimal holonomy algebra* is the smallest Lie algebra generated by curvature vector fields and by horizontal Berwald differentiation.

Proposition: $\mathfrak{R}_x(M)$ and $\mathfrak{hol}^*_x(M)$ are tangent to $\mathsf{Hol}_x(M)$.

Holonomy of Finsler surfaces ($\dim M = 2$)

Remarks:
- $\mathcal{I}_x \simeq \mathbb{S}^1$,
- $\mathsf{Hol}_x(M) \subset \mathsf{Diff}^\infty_+(\mathbb{S}^1)$
- $\dim \mathfrak{R}_x(M) \leq 1$,
- $\mathfrak{hol}^*_x(M)$ can be higher (even infinite) dimensional.

Proposition: *If* $\mathfrak{hol}^*_x(M)$ *contains* 4 *simultanously non-vanishing* $\mathbb{R}$*-linearly independent vector fields, then* $\mathsf{Hol}_x(M)$ *is not a finite-dimensional Lie group.*

Proof. S. Lie: If a finite-dimensional connected Lie group acts on a 1-dimensional manifold without fixed points, than its dimension is less than 4.

Some results: Finsler surfaces of non zero constant flag curvature

1. Tensorial characterization of the cases, when $\mathfrak{hol}^*(x) = \mathfrak{R}_x$,
2. Examples where $\mathfrak{hol}^*(x) = \mathfrak{R}_x$,
3. Examples where $\dim\big(\mathfrak{hol}^*(x)\big) = \infty$.

Finsler 2-manifolds with maximal holonomy group

Projectively flat Finsler spaces of constant curvature: $G^i(x,y)=\mathcal{P}(x,y)y^i$

Thm. If there exists $x_0 \in M$, where $\mathcal{F}(x_0,y) = \|y\|$ and $\mathcal{P}(x_0,y) = c\cdot\|y\|$, then

$$\overline{\mathsf{Hol}_{x_0}(M)} = \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1).$$

- $\mathcal{I}_{x_0} = \mathbb{S}^1$, $\mathsf{Hol}_0(M) \subset \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$
- $\mathfrak{hol}^*_0(M) \supset \mathsf{F}(\mathbb{S}^1) = \left\{\cos nt\,\frac{\partial}{\partial t}, \sin nt\,\frac{\partial}{\partial t}\right\}_{n\in\mathbb{N}}$ $\Rightarrow \overline{\mathsf{F}(\mathbb{S}^1)} = \overline{\mathfrak{hol}^*_0(M)} = \mathfrak{X}(\mathbb{S}^1)$
- $\exp(\mathfrak{X}(\mathbb{S}^1)) = \exp\left(\overline{\mathfrak{hol}^*_0(M)}\right) \subset \overline{\exp(\mathfrak{hol}^*_0(M))} \subset \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$
- $\Big\langle \exp(\mathfrak{X}(\mathbb{S}^1)) \Big\rangle \subset \Big\langle\, \overline{\exp(\mathfrak{hol}^*_0(M))}\, \Big\rangle \subset \overline{\mathsf{Hol}_0(M)} \subset \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$
- $\Big\langle \exp(\mathfrak{X}(\mathbb{S}^1)) \Big\rangle$ conj. inv. $\Rightarrow$ normal subgroup in $\mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$
- $\mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$ simple $\Rightarrow \big\langle \exp(\mathfrak{X}(\mathbb{S}^1)) \big\rangle = \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1) \Rightarrow \overline{\mathsf{Hol}_0(M)} = \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$

Corollary: The holonomy group of the Funk metric (constant negative curvature) and of the Bryant-Shen 2-spheres (constant positive curvature) are maximal.

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