Loops and geometry

Ágota Figula

University of Debrecen

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Preliminaries

Many questions in geometry need algebraic tools to solve them. One of these questions is the coordinatization of affine planes. Classical examples are the Desarguesian planes, they are coordinatized by skew fields. If the plane does not satisfy the Desargues' theorem, then the situation is changed.
The Moulton plane is an example of affine plane in which the Desargues' theorem does not hold. The points of the Moulton plane are the points in the real plane $\mathbb{R}^2$ and the lines are the regular lines as well with the exception that for lines with a negative slope, the slope doubles when they pass the y-axis. The ternary field $\mathcal{F}$ describing the incident structure of the corresponding projective plane is linear, its elements are the elements of the field $\mathbb{R}$, the addition is the usual addition in $\mathbb{R}$, the multiplication $*$ is the following new multiplication

$$m * x = \begin{matrix} \frac{1}{2}mx & m \leq 0, x \leq 0 \\ mx & m > 0 \text{ or } x > 0. \end{matrix}$$

In $(\mathbb{R}, +, *)$ the addition and the multiplication are commutative groups but the left and right distributive laws do not satisfy. Choosing $a < 0$, $b < 0$, $c > 0$ such that $b + c > 0$, then $a * (b + c) = a(b + c) = ab + ac$, but $a * b + a * c = \frac{1}{2}ab + ac$. In general, in the algebraic structure which coordinatizes an affine plane the addition and the multiplication are not a group, they are loops and between these operations none of the distributive laws must hold.

Definition

A loop $(L, \cdot)$ is a set L with a binary operation $\cdot$ such that the equations $a \cdot x = b$ and $y \cdot a = b$ are uniquely solvable for all given $a, b \in L$, i.e. $x = a \backslash b$, $y = b/a$ are uniquely determined, and there exists an element $1 \in L$ such that $1 \cdot x = x \cdot 1 = x$ for all $x \in L$. The mappings $\lambda_a : L \to L, x \mapsto a \cdot x$, $\rho_a : L \to L, x \mapsto x \cdot a$, $a \in L$, are called the left translations, respectively the right translations of L with respect to the element $a \in L$. All these maps are bijections of L.

Definition

A loop $(L, \cdot)$ is called topological, if L is a topological space and the operations $\cdot : L \times L \to L$, $\backslash : L \times L \to L$, $/ : L \times L \to L$ are continuous mappings.
For a topological loop the left translation maps and the right translation maps are homeomorphisms.

Affine planes and the algebraic structure coordinatizing them

We wish to deal with the class of quasifields. This includes the class of fields and the class of division algebras. Both of these algebraic structures are determined by the existence of both distributive laws. But already for these structures a complete classification without any additional assumption is impossible. The connected locally compact fields are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. The octonions $\mathbb{O}$ over $\mathbb{R}$ are the only non-associative locally compact alternative algebra.

Quasifields

Definition

A (left) quasifield is an algebraic structure $(Q, +, \cdot)$ *such that* $(Q, +)$ *is an abelian group with neutral element* 0, $(Q^* = Q \setminus \{0\}, \cdot)$ *is a loop with identity element* 1, *and between these operations the (left) distributive law* $x \cdot (y + z) = x \cdot y + x \cdot z$ *holds.*

A locally compact connected topological quasifield is a quasifield Q *such that the set* Q *is a locally compact connected topological space such that the addition, the multiplication, the left and right inversion are continuous. The mappings* λ_a, ρ_a *are homeomorphisms.*

Translation plane

The locally compact topological quasifields are the coordinate domain of locally compact translation planes.
If W is a line of an affine plane $\mathcal{A}$ and w denotes the improper point of W, then $W \cup \{w\}$ is a line of the projective extension of $\mathcal{A}$. The improper line of $\mathcal{A}$ is denoted by L_∞. A translation is a collineation such that its axis is the improper line L_∞ and it center is on L_∞. This means that a collineation is a translation if every line is parallel to its image and the invariant lines form a parallel class. This parallel class is called the direction of the translation. The translations of an affine plane build a group.

Definition

A translation plane is an affine plane $\mathcal{A}$ if the translation group of $\mathcal{A}$ operates transitively on the point set of $\mathcal{A}$.

Translation plane and quasifield

To a translation plane $\mathcal{A}$ there exists a quasifield $(Q, +, \cdot)$ such that the point set of $\mathcal{A}$ is

$$Q \times Q$$

and the lines of $\mathcal{A}$ are defined by equations

$$x = c,\ \ c \in Q \text{ and } y = m \cdot x + b,\ \ m, b \in Q.$$

The translations of $\mathcal{A}$ have the form $(x, y) \mapsto (x + u, y + v)$ with fixed $u, v \in Q$.

Definition

A subfield K of a left quasifield Q is called the kernel of Q if for all $k \in K$ and $x, y \in Q$ one has

$$(x + y) \cdot k = x \cdot k + y \cdot k \text{ and } (x \cdot y) \cdot k = x \cdot (y \cdot k).$$

Proposition

The left quasifield Q is a right vector space over K. Moreover, for all $a \in Q$ the map $\lambda_a : Q \to Q, x \mapsto a \cdot x$ is K-linear.

Vector space structure of locally compact connected quasifields

The kernel K is the field of the real numbers and the dimension of Q over $\mathbb{R}$ is 1, 2, 4 or 8.
The kernel K is the field of the complex numbers and the dimension of Q over $\mathbb{C}$ is 1 or 2.
Q is the skew field of quaternions.

We deal with locally compact 2-dimensional topological connected quasifields which coordinatize 4-dimensional locally compact non-Desarguesian topological translation planes $\mathcal{A}$. Their kernel is $\mathbb{R}$, $(Q,+)$ is the vector group $\mathbb{R}^2$ and the multiplicative loop $(Q\backslash\{0\},\cdot)$ is homeomorphic to $\mathbb{R}\times S^1$, where S^1 is the circle.

2-dimensional locally compact connected topological quasifields

Let e_1 be the identity element of the multiplicative loop $Q^* = (Q\setminus\{0\},\cdot)$ of Q, which generates the kernel $K=\mathbb{R}$ of Q as a vector space and let $B=\{e_1,e_2\}$ be a basis of the right vector space Q over K. Once we fix B, we identify Q with the vector space of pairs $(x,y)^t\in\mathbb{R}^2$ and K with the subspace of pairs $(x,0)^t$. The element $(1,0)^t$ is the identity element of the multiplicative loop Q^* of Q. Moreover, the $\mathbb{R}$-linear maps λ_a can be written as 2×2 matrices over $\mathbb{R}$.

Since the set $\Lambda = \{\lambda_a, a \in Q^*\}$ consists of permutations of Q^* such that for λ_a, λ_b, $a \neq b$, the permutation $\lambda_a \lambda_b^{-1}$ is fixed point free. For matrices this means that for all $z \in Q^*$

$$z \neq z\lambda_a\lambda_b^{-1} \Leftrightarrow 0 \neq z(\lambda_a - \lambda_b) \Leftrightarrow 0 \neq \det(\lambda_a - \lambda_b).$$

Hence the multiplicative loop Q^* of a 2-dimensional locally compact connected topological quasifield Q can be equivalently given by the set Λ of 2×2 matrices over $\mathbb{R}$ such that for any $A, B \in \Lambda$, $A \neq B$, $\det(A - B) \neq 0$.
Let A be a 2×2 real matrix and $W = \mathbb{R}^4$ vector space. Define the 2-dimensional subspace $U_A = \{(x, Ax); x \in \mathbb{R}^2\}$ of W and $U_\infty = \{(0, x), x \in \mathbb{R}^2\}$.

Put $\widetilde{\Lambda} = \{U_A, A \in \Lambda\} \cup \{U_\infty\}$, where Λ is the set of left multiplications of Q. Then,
(i) for any $A \neq B \in \Lambda$, $U_A \cap U_B = U_A \cap U_\infty = \{(0,0)\}$ and
(ii) for any nonzero element $(x, y) \in W$ there is a unique element of $\widetilde{\Lambda}$ containing it. These two properties say that $\widetilde{\Lambda}$ form a partition of W into 2-dimensional subspaces. Such partitions are called spreads. Conversely, any 2-dimensional spread $\widetilde{\Lambda}$ of $W = \mathbb{R}^4$ defines a 2-dimensional quasifield Q and a translation plane $\mathcal{A}$ which are coordinatized by Q.

The points of $\mathcal{A}$ are the elements of W and the lines of $\mathcal{A}$ are the sets $\{U + x | U \in \mathcal{B}, x \in W\}$.

Using 2-dimensional spreads D. Betten has classified all locally compact 4-dimensional translation planes which admit an at least 7-dimensional collineation group.

Example: Let f be a continuous, non-linear, strictly monotone function defined for $0 \leq r \in \mathbb{R}$ with $f(0) = 0$ and $\lim_{r \to \infty} f(r) = \infty$. The 2-dimensional normalized spreads defined by

$$(1) \qquad S = \{(0, x), x \in \mathbb{R}^2\} \cup \{(x, A_{(r,\varphi)}x), x \in \mathbb{R}^2\}$$

with $A_{(r,\varphi)} = \left\{ \begin{pmatrix} r\cos\varphi & -\frac{f(r)\sin\varphi}{f(1)} \\ r\sin\varphi & \frac{f(r)\cos\varphi}{f(1)} \end{pmatrix}, r > 0,\ \varphi \in [0, 2\pi) \right\}$
determines a 4-dimensional locally compact non-desarguesian planes $\mathcal{A}_f$.

P. Nagy and K. Strambach: Loops in Group Theory and Lie Theory, Gruyter, 2002, has been proved that

Proposition

The group topologically generated by the left translations of Q^* is the connected component of $GL_2(\mathbb{R})$.

Using loop theory we can give a matrix representation for the left translations $\lambda_a, a \in Q^*$ of Q^* in the following way: Consider the group $G = GL_2^+(\mathbb{R})$ generated by the left translations of Q^*. G is a topological transformation group acting transitively and effectively on the set Q^*. Since $dim(G) = 4$ and dim $Q^* = 2$ there is a 2-dimensional subgroup H of G which leaves fixed the identity element $e \in Q^*$. This called the stabilizer H of $e \in Q^*$. H is a subgroup of G such that H does not contain any non-trivial normal subgroup of G.

In our case we may assume that H is the subgroup $\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$ isomorphic to $\mathcal{L}_2$.
G/H denotes the set of left cosets $xH, x \in G$. The set $\Lambda = \{\lambda_a, a \in Q^*\}$ acts sharply transitively on Q^* and Λ is the image of a sharply transitive continuous section $\sigma : G/H \to G$ with the properties: $\sigma(H) = 1 \in G$, $\sigma(G/H)$ generates G and $\sigma(G/H)$ acts sharply transitively on G/H, i.e. to any xH and yH there exists precisely one $z \in \sigma(G/H)$ with $zxH = yH$, because for every such continuous section $\sigma : G/H \to G$ the multiplication given by $xH * yH = \sigma(xH)yH$ on the factor space G/H yields a loop $L(\sigma)$ having $\sigma(G/H)$ as the set Λ of the left translations of $L(\sigma)$.

The elements g of $G = GL_2^+(\mathbb{R})$ have a unique decomposition as the product

$$g = \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}$$

with suitable elements $u > 0,\ k > 0,\ l \in \mathbb{R},\ t \in [0, 2\pi)$.
Hence the loop Q^* homeomorphic to $\mathbb{R} \times S^1$ corresponds to a continuous section $\sigma : G/H \to G$;

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \cdot \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix}, \tag{2}$$

where the pair of continuous functions $a(u,t), b(u,t) : \mathbb{R}_{>0} \times [0, 2\pi) \to \mathbb{R}$, where $\mathbb{R}_{>0}$ is the set of positive numbers, satisfies the following conditions:

$$a(u,t) > 0, \quad a(1,0) = 1, \quad b(1,0) = 0.$$

As Q is a left quasifield, any $(x,y)^t \in Q^*$ induces a linear transformation $M(x,y) \in \sigma(G/H)$. More precisely

$$\textbf{(3)} \quad \begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} = M_{(x,y)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r\cos\varphi & r\sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix} \cdot \begin{pmatrix} a(r,\varphi) & b(r,\varphi) \\ 0 & a^{-1}(r,\varphi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $x = r\cos(\varphi)a(r,\varphi)$, $y = -r\sin(\varphi)a(r,\varphi)$.

As the kernel K consists of pairs $(0,0)^t$ and $(r\cos(k\pi)a(r,k\pi),0)^t$, $r>0$, $k\in\{0,1\}$, the left translation maps with respect to the elements of K have the form

$$M(r)=\begin{pmatrix} r\cos(k\pi)a(r,k\pi) & r\cos(k\pi)b(r,k\pi) \\ 0 & r\cos(k\pi)a^{-1}(r,k\pi) \end{pmatrix}. \tag{4}$$

As the identity $e\in Q^*$ is $(1,0)^T$ the corresponding matrix is I. Since the set of vectors $(x,y)^t, x,y\in\mathbb{R}$ consists of all elements of Q, there exists a unique left translation λ_a such that its representation as 2×2 matrix has $(x,y)^t$ as the first column. As to each real number $r\cos(k\pi)a(r,k\pi)$ belongs precisely one matrix $M(r)$ of form (4), the functions $f_1(r)=ra(r,0)$, $f_2(r)=-ra(r,\pi)$ are strictly monotone.
Since $K=\mathbb{R}$ the group generated by all left translations of the positive elements of $\mathbb{R}_>$ of K is isomorphic to the multiplicative group of the field $\mathbb{R}$. Therefore, the function $a(r,0)$ is a homomorphism and the identity
$b(r_1r_2,0)=a(r_1,0)b(r_2,0)+b(r_1,0)a^{-1}(r_2,0)$ is satisfied for all

The section σ given by (2) is sharply transitive precisely if for all pairs (u_1, t_1), (u_2, t_2) in $\mathbb{R}_{>0} \times [0, 2\pi)$ there exists precisely one $(u, t) \in \mathbb{R}_{>0} \times [0, 2\pi)$ and $k > 0$, $l \in \mathbb{R}$ such that

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \cdot \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix} \cdot \begin{pmatrix} u_1\cos t_1 & u_1\sin t_1 \\ -u_1\sin t_1 & u_1\cos t_1 \end{pmatrix} =$$

$$\begin{pmatrix} u_2\cos t_2 & u_2\sin t_2 \\ -u_2\sin t_2 & u_2\cos t_2 \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}. \tag{5}$$

We get that $u = u_1^{-1} u_2$. Therefore the system (5) of equations is uniquely solvable if and only if for any fixed $u > 0$ the mapping

(6)

$$\sigma_u : \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} H \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix}$$

determines a loop F_u homeomorphic to S^1. This is the case if and only if $b(u, 0) = 0$ for all $u > 0$.

Proposition

Any locally compact 2-dimensional semifield (i.e. in Q also the right distributive law holds) is the field of complex numbers.

(cf. P. Plaumann and K. Strambach: Zweidimensionale Quasialgebren mit Nullteilern. Aequationes Math., Volume 15, 1977, pp 249-264.)

Our aim is to characterize the algebraic structure of the multiplicative loops for locally compact 2-dimensional topological connected quasifields. In particular, our attention turns to multiplicative loops which have either a normal subloop of positive dimension or which contain a 1-dimensional compact subgroup.

Definition

The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L. We call a locally compact connected loop quasi-simple if it contains no normal subloop of positive dimension.

Proposition

The multiplicative loop Q^* of a 2-dimensional locally compact quasifield Q contains for any $u > 0$ a 1-dimensional compact subloop.

The image of the section

(7) $\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix},$

$u > 0$, $t \in [0, 2\pi)$ with $b(u,0) = 0$ for all $u > 0$ acts sharply transitively on the point set $\mathbb{R}^2 \setminus \{(0,0)^t\}$. The subgroup $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, u > 0 \right\}$ leaves any line through $(0,0)^t$ fixed, the subset

(8)
$$\mathcal{T} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix}, t \in [0, 2\pi) \right\}$$

acts sharply transitively on the oriented lines through $(0,0)^t$ for any $u > 0$. Therefore $\mathcal{T}$ corresponds to a 1-dimensional compact loop T.

Proposition

A 2-dimensional locally compact quasifield Q is the field of complex numbers if and only if the multiplicative loop Q^* contains a 1-dimensional compact normal subloop and the set $\mathcal{T}$ is a normal subset in the set of all left translations of Q^*.

If Q is the field of complex numbers, then Q^* is the group $SO_2(\mathbb{R}) \times \mathbb{R}$ and the assertion holds.
The set $\mathcal{T}$ is a normal subset in the set of all left translations of Q^* precisely if $\mathcal{T}$ is isomorphic to the group $SO_2(\mathbb{R})$. But the compact group $SO_2(\mathbb{R})$ is not normal in $GL_2^+(\mathbb{R})$. Hence Q^* is not a proper loop and the assertion follows.

Proposition

If the multiplicative proper loop Q^* of a 2-dimensional locally compact connected topological quasifield Q is not quasi-simple, then the set $\mathcal{K}$ of the left translations of Q^* belonging to the kernel K of Q has the form $\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$, which is a normal subgroup of the set Λ_{Q^*} of all left translations of Q^*.

If the loop Q^* is proper and not quasi-simple, then the left translations of a normal subloop N of Q^* generate a normal subgroup $\mathcal{N}$ of $GL_2^+(\mathbb{R})$ which is the group topologically generated by all left translations of Q^*. From the above Proposition N is non-compact. Then N is a line (homeomorphic to $\mathbb{R}$). If N is proper loop, then the group generated by the left translations of N is the universal covering of $\widetilde{PSL_2}(\mathbb{R})$. But this group has no linear representation. Hence N is a group and $\mathcal{N}$ is isomorphic to N. The only possibility for $\mathcal{N}$ is the centre of $GL_2^+(\mathbb{R})$. The assertion follows.

The only possibility for a normal subloop N of positive dimension of the proper loop Q^* having $(1,0)^t$ as identity is the group $N := \{(s,0)^t, s \in \mathbb{R} \setminus \{0\}\}$.

Theorem

The multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is proper and not quasi-simple if and only if for all $r > 0$, $\varphi \in [0, 2\pi)$, $k \in \{0,1\}$ one has $a(r, k\pi) = 1$, $b(r, k\pi) = 0$, $a(r, \varphi + k\pi) = a(1, \varphi)$ and $b(r, \varphi + k\pi) = b(1, \varphi)$. Then Q^* is a central extension of a 1-dimensional normal subgroup N isomorphic to $\mathbb{R} \times Z_2$, where Z_2 is the group of order 2, by a subloop homeomorphic to the 1-sphere.

Corollary

The multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is the direct product of the group $\mathbb{R}$ and a subloop homeomorphic to the 1-sphere if and only if Q is the field of complex numbers.

Definition

We call the multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q decomposable, *if the set of all left translations of Q^* is a product $\mathcal{T}\mathcal{K}$ with $|\mathcal{T} \cap \mathcal{K}| \leq 2$, where $\mathcal{T}$ is the set of all left translations of a 1-dimensional compact subloop and $\mathcal{K}$ is the set of all left translations corresponding to the kernel K of Q.*

Theorem

The multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is decomposable if and only if for all $r > 0$, $t \in [0, 2\pi)$, $k \in \{0, 1\}$ one has

$$a(r, t + k\pi) = a(1, t)a(r, k\pi),$$

$$b(r, t + k\pi) = a(1, t)b(r, k\pi) + a^{-1}(r, k\pi)b(1, t).$$

Theorem

If the multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is not quasi-simple, then Q^* is decomposable.

The set Λ_{Q^*} of the left translations of Q^* with a normal subloop of positive dimension and with $(1,0)^t$ as identity can be written into the form

(9)

$$\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(1,t) & ub(1,t) \\ 0 & ua^{-1}(1,t) \end{pmatrix}, u > 0, t \in [0, 2\pi) \right\}$$

with $a(1, k\pi) = 1,\ b(1, k\pi) = 0,\ k \in \{0, 1\}$.

Proposition

The set Λ_{Q^*} of all left translations of the multiplicative loop Q^* for a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* contains the group $\mathrm{SO}_2(\mathbb{R})$ if and only if Λ_{Q^*} has the form

$$(10) \qquad \Lambda_{Q^*} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(u,0) & ub(u,0) \\ 0 & ua^{-1}(u,0) \end{pmatrix} \right\}$$

$u > 0$, $t \in [0, 2\pi)$ where $a(u,0)$, $b(u,0)$ are arbitrary continuous functions with $a(u,0) > 0$ such that $ua(u,0)$ is strictly monotone. In this case Q^* is decomposable.

Corollary

If the multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q has a normal subloop of positive dimension or if it contains the group $\mathrm{SO}_2(\mathbb{R})$, then Q^* is decomposable.

Results about differentiable multiplicative loops

Using Fourier series we classified the C^1-differentiable multiplicative loops Q^* of 2-dimensional locally compact quasifields Q. We showed that any 1-dimensional C^1-differentiable compact loop is a factor of a decomposable multiplicative loop of a locally compact connected quasifield coordinatizing a 4-dimensional translation plane.

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H. Salzmann, D. Betten, T. Grundhoefer, H. Haehl, M. Stroppel, Compact projective planes, Gruyter, 1995.

D. Betten has classified all locally compact 4-dimensional translation planes which admit an at least 7-dimensional collineation group. Using his classification we determined the multiplicative loops Q^* of the quasifields which coordinatize the 4-dimensional non-desarguesian translation planes $\mathcal{A}$ admitting an at least seven-dimensional collineation group and we studied their properties. Mostly we get that the loop Q^* is quasi-simple and non-decomposable. If the group generated by the left translations of a loop L is simple, then L is also simple. The multiplicative loops Q^* of 2-dimensional locally compact quasifields show that there are many interesting 2-dimensional locally compact quasi-simple loops for which the group generated by their left translations has a one-dimensional centre.

Theorem

Let $\mathcal{A}$ be a 4-dimensional locally compact non-desarguesian translation plane which admits an at least 7-dimensional collineation group Γ. Then the multiplicative loop Q^* of the quasifield Q which coordinatizes $\mathcal{A}$ is decomposable if and only if one of the following cases occurs:
(a) Γ is 8-dimensional, the translation complement C is the group $\mathrm{GL}_2(\mathbb{R})$ and acts reducibly on the translation group $\mathbb{R}^4$;
(b) Γ is 7-dimensional, the translation complement C fixes two distinct lines $\{S, W\}$ through the origin and leaves on one of them, one or two 1-dimensional subspaces invariant;
(c) Γ is 7-dimensional, the translation complement C fixes two distinct lines $\{S, W\}$ through the origin and acts transitively on the spaces P_S and P_W but does not act transitively on the product space $P_S \times P_W$, where P_S and P_W are the sets of all 1-dimensional subspaces of S, respectively of W.

In case a) one obtains the one-parameter family $\mathcal{A}_w$, $w > 1$, of the non-desarguesian translation planes corresponding to the following spreads:

$$\{S\} \cup \left\{ \begin{pmatrix} s & -v \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v \geq 0 \right\} \cup \left\{ \begin{pmatrix} s & \frac{-v}{w} \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v < 0 \right\},$$

$w > 1$ (cf. Betten: 4-dimensionale Translationsebenen. Math. Z.). Λ_{Q^*} has the form

(11)

$$\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ra(1,t) & rb(1,t) \\ 0 & ra^{-1}(1,t) \end{pmatrix}, r > 0, t \in [0, 2\pi) \right\}$$

with $a(1,t) = 1$ and $b(1,t) = 0$ for $0 \leq t \leq \pi$, $a(1,t) = 1/\sqrt{\cos^2 t + \frac{\sin^2 t}{w}}$ and $b(1,t) = a(1,t)\frac{1-w}{w} \sin t \cos t$ for $\pi < t < 2\pi$. The multiplicative loop Q_w^* is decomposable and a central extension of the normal subgroup $\widetilde{N^*} \cong \mathbb{R}$ corresponding to the connected component of $\widetilde{\mathcal{K}} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$ with a subloop homeomorphic to the 1-sphere.

In case (b) the group C fixes either two 1-dimensional subspaces of S, or only one 1-dimensional subspace of S. Then in the first case one obtains a family of translation planes corresponding to the normalized spreads

(12)

$$\{S, W\} \cup \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} s & c(s^w - s) \\ 0 & s^w \end{pmatrix},\ s, \varphi \in \mathbb{R}, s > 0 \right\},$$

in the second case

$$(13) \quad \{S, W\} \cup \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} e^s & e^s \frac{s}{d} \\ 0 & e^s \end{pmatrix}, s, \varphi \in \mathbb{R} \right\},$$

(cf. Betten, 4-dimensionale Translationsebenen mit 7-dimensionaler Kollineationsgruppe. J. Reine Angew. Math.).

In both cases these spreads coincide with the set $\Lambda = SO_2(\mathbb{R})\mathcal{K}$ given by

$$\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ra(r,0) & rb(r,0) \\ 0 & ra^{-1}(r,0) \end{pmatrix}, r > 0, t \in [0, 2\pi) \right\}$$

such that the set $\mathcal{K}$ corresponding to the kernel K_r of Q is determined by the functions $a(r,0) = r^{\frac{1-w}{1+w}}$, $b(r,0) = c(r^{\frac{w-1}{w+1}} - r^{\frac{1-w}{1+w}})$ with $r = s^{\frac{w+1}{2}}$, $s > 0$, $t = -\varphi$, where s and φ are variables of the first spreads and $a(r,0) = 1$, $b(r,0) = \frac{\ln r}{d}$, with $r = e^s$, $t = -\varphi$, where s and φ are variables of the second spreads. In the first case the quasifields $Q_{w,c}$ coordinatize a family of planes $\mathcal{A}_{w,c}$ such that for the parameters $w \neq 1, c$ one has $0 < w$ and $(w-1)^2c^2 \leq 4w$. In the second case the quasifields Q_d coordinatize a one-parameter family of planes $\mathcal{A}_d$ such that $4d^2 \geq 1$. In both cases Q^* is decomposable and contains the group $SO_2(\mathbb{R})$.

In case (c) there is a family of translation planes which correspond to the normalized spreads

$$\{S, W\} \cup \left\{ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} a_{11}(t) & -\frac{c}{d} a_{11}(t) + \frac{1}{d} a_{21}(t) \\ a_{12}(t) & -\frac{c}{d} a_{12}(t) + \frac{1}{d} a_{22}(t) \end{pmatrix}, s \geq 0, t \in \mathbb{R} \right\}$$

with $a_{11}(t) = \cos nt \cos mt + c \sin nt \cos mt + d \sin nt \sin mt$,
$a_{12}(t) = d \sin nt \cos mt - \cos nt \sin mt - c \sin nt \sin mt$,
$a_{21}(t) = d \cos nt \sin mt - \sin nt \cos mt + c \cos nt \sin mt$,
$a_{22}(t) = d \cos nt \cos mt + \sin nt \sin mt - c \cos nt \sin mt$ (cf. Betten, 4-dimensionale Translationsebenen mit 7-dimensionaler Kollineationsgruppe. J. Reine Angew. Math.).

These spreads coincide with the set Λ_{Q^*} having the form

(14)

$$\left\{ \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} \begin{pmatrix} ra(1,u) & rb(1,u) \\ 0 & ra^{-1}(1,u) \end{pmatrix}, r > 0, u \in [0, 2\pi) \right\}$$

with the periodic functions $a(1,u)$ and $b(1,u)$

$$a(1,u) = \sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}, \; b(1,u) = \frac{\sin nt \cos nt (d^2 - 1 - c^2) - c \sin^2 nt (d^2 + 1 + c^2)}{d\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}$$

such that

$$r \cos u = \frac{s(\cos nt \cos mt + c \sin nt \cos mt + d \sin nt \sin mt)}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}},$$

$$r \sin u = \frac{s(d \sin nt \cos mt - \cos nt \sin mt - c \sin nt \sin mt)}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}$$

and $s \geq 0$.

The quasifields $Q_{m,n,c,d}$ coordinatize a family of planes $\mathcal{A}_{m,n,c,d}$ such that the parameters $m, n \in \mathbb{Z}$, $(m, n) = 1$, $c, d \in \mathbb{R}$ satisfy the conditions

$$\begin{array}{lll} m = n = 1 & & \text{and} \quad -1 \leq d < 0 \\ m = 1, 2, 3, \cdots & n = m + 1 & \text{and} \quad d > 0 \\ m = 1, 3, 5, \cdots & n = m + 2 & \text{and} \quad d > 0 \end{array}$$

$$(n - m)^2 B \geq (n + m)^2 A, \text{ where } A = \frac{(d-1)^2 + c^2}{4d} \text{ and } B = \frac{(d+1)^2 + c^2}{4d}.$$

The loops $Q^*_{m,n,c,d}$ are split extensions of the normal subgroup $\widetilde{N^*} \cong \mathbb{R}$ corresponding to the connected component of $\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$ with a subloop homeomorphic to the 1-sphere.

Hyperbolic space loop

The vector addition in the hyperbolic space yields loop, which is called hyperbolic space loop. The loop operation is given by

$$x \cdot y = \frac{x + y}{1 + \langle x, y \rangle},$$

and the identity $1 \in L$ is the zero vector.

A. Ungar, Analytic hyperbolic geometry and Albert Einstein's special theory of relativity, World Scientific Publisher, 2008.
L. Sbitneva, Nonassociative geometry of special relativity, International Journal of Theoretical Physics, Vol. 40, 2001.

Latin square

If L is a finite set, then L is a normalized Latin square.

For example:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 & 4 \end{array}$$

$$i * j = a_{ij}$$

$$(2*4)*3 = 5*3 = 2,\ 2*(4*3) = 2*5 = 3,$$

$$(2*2)*2 = 4*2 = 1,\ 2*(2*2) = 2*4 = 5$$

If the translation plane $\mathcal{A}$ coordinatizing by Q is non-Desarguesian, then either the vector space Q is a real vector space and has dimension $m = 2^k$, $k = 1, 2, 3$, or the vector space Q is a complex vector space and its topological dimension $m = 4$. The translation plane $\mathcal{A}$ has dimension $2m$.
The multiplicative loop Q^* is homeomorphic to $\mathbb{R} \times S^n$, where S^n is an n-sphere with $n \in \{1, 3, 7\}$ and the group G topologically generated by the left translations of Q^* is a closed connected subgroup of the group $GL_{n+1}(\mathbb{R})$ if $K = \mathbb{R}$ or G is a closed connected subgroup of the group $GL_2(\mathbb{C})$ if $K = \mathbb{C}$.

Definition

Let W be a vector space over a skew field. A collection $\mathcal{B}$ of subspaces of W with $|\mathcal{B}| \geq 3$ is called a spread of W if for any two different elements U_1, $U_2 \in \mathcal{B}$ we have $W = U_1 \oplus U_2$ and every vector of W is contained in precisely one element of $\mathcal{B}$.

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