On the Structure of Chevalley Groups

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The Big Picture

ODE → Lie-group ⇔ Lie-algebra

Simple groups (of Lie-type) ↔ Chevalley correspondence ↔ Simple Lie-algebras

? ↔ Simple groups

Finite State Machines
Simple Lie algebras over complex field $\mathbb{C}$

- An $I$ subalgebra of the Lie algebra $L$ is called **ideal** if $[x, i] \in I$ for all $i \in I$ and $x \in L$.

- A Lie algebra $L$ is **simple** if does not contain any non-trivial ideal.

Let $L$ be a **finite dimensional** simple Lie algebra over $\mathbb{C}$. Then there exist a **unique** (up to isomorphism) **root space** or Cartan decomposition such that

$$L = H \oplus \bigoplus_{r \in \Phi} L_r$$

where $H$ is a **Cartan subalgebra** and each **root space** $L_r$ is one dimensional.

- The **roots** $r$ can be represented as vectors of the **real** Euclidean space.

$$\dim H = |\Phi|$$
Fundamental bases

There exist a fundamental base $\Pi \subset \Phi$ such that

- the elements of $\Pi$ are linearly independent.
- Every element of $\Phi$ is a linear combination of $\Pi$ such that all multipliers are integer and all of them are positive or all of them are negative.
- $|\Pi| = l$ is called the rank of the root system $\Phi$. This equals to the rank of the Lie algebra $L$.

There are only 4 cases of a root system rank of 2. Evenmore, for $\alpha, \beta \in \Phi$ ($\beta \neq \pm \alpha$)

$$\{C\alpha + C\beta\} \cap \Phi$$

is the one of the following four vector systems:
Root spaces of dimension 2
Dynkin diagrams

- Let $\Pi$ be a fundamental base of elements $I$.
- The vertices of the Dynkin diagram are the elements of $\Pi$ (Note that all fundamental bases are isomorphic).
- Let $\alpha, \beta \in \Pi$ and $\theta$ be the angle between $\alpha$ and $\beta$. The number of edges between $\alpha$ and $\beta$ is $4 \cos^2 \theta$.
- Why $4 \cos^2 \theta$ is integer?
- If $|\alpha| \neq |\beta|$ then there is an arrow on the edges directed to the shorter vector.
Infinite series of Dynkin diagrams

\[ A_l \ (l \geq 1) : \quad 1 \rightsquigarrow 2 \rightsquigarrow 3 \cdots \rightsquigarrow l-1 \rightsquigarrow l \]

\[ B_l \ (l \geq 2) : \quad 1 \rightsquigarrow 2 \rightsquigarrow \cdots \rightsquigarrow l-2 \rightsquigarrow l-1 \rightsquigarrow l \]

\[ C_l \ (l \geq 3) : \quad 1 \rightsquigarrow 2 \rightsquigarrow \cdots \rightsquigarrow l-2 \rightsquigarrow l-1 \rightsquigarrow l \]

\[ D_l \ (l \geq 4) : \quad 1 \rightsquigarrow 2 \rightsquigarrow \cdots \rightsquigarrow l-3 \rightsquigarrow l-2 \rightsquigarrow l-1 \rightsquigarrow l \]
 Infinite series of Dynkin diagrams

\[ E_6 : \]
\[ E_7 : \]
\[ E_8 : \]
\[ F_4 : \]
\[ G_2 : \]
The structure constants of simple Lie algebras

Let $L$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. Take a Cartan decomposition such that

$$L = H \oplus \bigoplus_{r \in \Phi} L_r$$

Here one can choose elements $e_r \in L_r$, $e_{-r} \in L_{-r}$, $h_r \in H$ for each $r \in \Phi$ such that

- $[h_r, e_r] = 2e_r$, $[h_r, e_{-r}] = 2e_{-r}$ and $[e_r, e_{-r}] = h_r$. This means that for each $r \in \Phi$ the linear combinations of $e_r, e_{-r}, h_r \in H$ form a simple Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$

- $[h_s, e_r] = A_{rs}e_r$ and $[e_r, e_s] = N_{rs}e_{r+s}$ ($r + s \neq 0$). These constants $A_{rs}$ and $N_{rs}$ are called structure constants.
Simple Lie algebras over arbitrary field

**Theorem (Chevalley)**

*The basis elements* $e_r \in L_r$, $e_{-r} \in L_{-r}$ *and* $h_r \in H$ *(for all* $r \in \Phi$) *can be chosen such that all structure constants* $A_{rs}$, $N_{rs}$ *are integers.*

Why this result is so essential?
Because one can define the simple Lie algebra $L = L(\mathbb{C})$ over $\mathbb{C}$ in an abstract way using the calculation rules:

**Definition**

- Let $\{e_r, e_{-r}, h_r | r \in \Phi\}$ *is a basis of* $L$ *(over* $\mathbb{C}$).
- $[h_r, e_r] = 2e_r$, $[h_r, e_{-r}] = 2e_{-r}$ *and* $[e_r, e_{-r}] = h_r$.
- $[h_s, e_r] = A_{rs}e_r$ *and* $[e_r, e_s] = N_{rs}e_{r+s}$ *(for* $r + s \neq 0$).
Simple Lie algebras over arbitrary field

Since all multipliers are integers in the previous definition the complex field can change to an arbitrary other field $K$. Therefore the definition of the simple Lie algebra $L(K)$ over an arbitrary field $K$:

\begin{itemize}
  \item Let $\{e_r, e_{-r}, h_r \mid r \in \Phi\}$ is a basis of $L(K)$ (over $K$).
  \item $[h_r, e_r] = 2e_r$, $[h_r, e_{-r}] = 2e_{-r}$ and $[e_r, e_{-r}] = h_r$.
  \item $[h_s, e_r] = A_{rs}e_r$ and $[e_r, e_s] = N_{rs}e_{r+s} \ (r + s \neq 0)$.
\end{itemize}
The automorphisms of Lie algebras

- An isomorphism \( \phi : L \to L \) is called an automorphism of \( L \).
- All automorphisms of \( L \) form a group denoted by \( \text{Aut}(L) \).
- The adjoint map \( \text{ad}_x : L \to L \) such that \( y \to [x, y] \) is a nilpotent derivation if \( x = \lambda e_r \) for each \( r \in \Phi, \lambda \in K \).
- If \( \delta \) is a nilpotent derivation of \( L \) \( (\delta^n = 0) \) then the exponential map:

\[
\exp(\delta) = \text{id} + \delta + \frac{\delta^2}{2!} + \cdots + \frac{\delta^n}{n!}
\]

is an automorphism of \( L \).
The automorphisms of Lie algebras

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- If $\delta$ is a nilpotent derivation of $L$ ($\delta^n = 0$) then the exponential map:

$$ \exp(\delta) = \text{id} + \delta + \frac{\delta^2}{2!} + \cdots + \frac{\delta^n}{n!} $$

is an automorphism of $L$.
The definition of Chevalley groups

Definition

The Chevalley group associated to the simple Lie algebra $L(K)$ is the $GL(K)$ subgroup of $\text{Aut}(L)$ such that

$$GL(K) = \langle \exp(\text{ad}\lambda e_r) | r \in \Phi, \lambda \in K \rangle$$

- Depending on the choice of the field $K$ ($K = \mathbb{R}$ or $K = \mathbb{Q}$) a Chevalley group can be infinite or finite (if $K$ is a finite field).
- The Chevalley groups are simple (except a few cases) but the proof is not simple.
The structure of Chevalley groups

- Let \( x_r(\lambda) = \exp(\text{ad}\lambda e_r) \). The Chevalley group \( G_L(K) \) is generated by all \( x_r(\lambda), r \in \Phi, \lambda \in K \).
- The root subgroup \( X_r = \langle x_r(\lambda) | \lambda \in K \rangle \)
- The subgroup \( \langle X_r, X_{-r} \rangle \) is a homomorph image of \( SL_2(K) \) such that
  \[
  \phi : \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \rightarrow x_r(\lambda)
  \]
  \[
  \phi : \begin{pmatrix} 1 & 0 \\ \lambda & 0 \end{pmatrix} \rightarrow x_{-r}(\lambda)
  \]

Why is this important to find homomorph images of \( SL_2(K) \) in the Chevalley group \( G_L(K) \)? Because, it is easy to calculate using matrices of \( 2 \times 2 \).
(B,N)-pairs

Let \( h_r(\lambda) = \phi \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \), \( n_r(\lambda) = \phi \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} \)

Let \( B = \langle X_r | r \in \Phi \rangle \)

Let \( N = \langle n_r(\lambda), h_r(\lambda) | r \in \Phi, \lambda \in K \rangle \)

In general if group has \((B,N)\)-pair, so there exists subgroups \( B, N \) with nice properties then we have an argument to prove the simplicity of the group.
Properties (B,N)-pairs

Let $B, N$ be subgroups of $G$

- $G$ is generated by $B$ and $N$
- $B \cap N$ is normal subgroup in $N$
- The group $W = N/B/B \cap N$ is generated by a set of element $w_i \ i \in I$ such that $w_i^2 = 1$
- If $n_i \in N$ maps to $w_i$ under the natural homomorphism of $N$ into $W$, and if $n$ is any element of $N$, then

$$Bn_iB \cdot BnB \subseteq Bn_iNB \cup BnB.$$  

- $n_iBn_i \not\subseteq B$