Finsler 2-manifolds with maximal holonomy group of infinite dimension

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Parallel translation, holonomy

- $M$ is simply connected
- Finslerian metric: $g = g_{ij}(x, y)dx^i \otimes dx^j$
- Geodesics: $\ddot{x}^i + 2G^i(x, \dot{x}) = 0, \quad G^i := \frac{1}{4}g^{il}\left(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)y^jy^k$.
- Parallel vector field $X(t)$ along a curve $c(t)$:
  \[ \nabla_{\dot{c}}X(t) = \left(\frac{dX^i(t)}{dt} + \Gamma^i_j(c(t), X(t))\dot{c}^j(t)\right)\frac{\partial}{\partial x^i} = 0, \quad \Gamma^i_j = \frac{\partial G^i}{\partial y^j}. \]
- Parallel translation along a curve $c : [0, 1] \to M$:
  \[ \tau_c : T_{c_0}M \to T_{c_1}M, \quad \Rightarrow \begin{cases} \tau(\lambda v) = \lambda \tau(v) \\ \|\tau(v)\| = \|v\| \end{cases} \Rightarrow \tau_c : \mathcal{I}_{c_0} \to \mathcal{I}_{c_1} \]
- The holonomy group is generated by parallel translation along closed curves
  \[ \Rightarrow \text{subgroup of } \text{Diff}_+^{\infty}(\mathcal{I}_x) \text{ determined by parallel translations.} \]
Parallel translation: geometric construction

\[ \tau(M) \]

\[ \tau(v) = w \]
• $R \equiv 0$

• $R \neq 0$
Tangent Lie algebras to a subgroup \( H \) of \( \text{Diff}^\infty(\mathcal{I}) \)

**Def:** • A vector field \( X \) is *tangent* to \( H \), if there exists a differentiable curve of diffeomorphisms \( \{\phi_t\} \) in \( H \) such that

\[
\phi_0 = \text{Id}, \quad \frac{\partial \phi_t}{\partial t} \bigg|_{t=0} = X.
\]

• A Lie subalgebra \( \mathfrak{h} \) of \( \mathfrak{X}^\infty(\mathcal{I}) \) is called *tangent* to \( H \), if all elements of \( \mathfrak{h} \) are tangent to \( H \).

\[ \mathfrak{h} \text{ tangent to } H \implies \text{information on } H \]

**Property:** If \( \mathfrak{h} \) is tangent to a closed subgroup \( H \), then

\[
\exp(\mathfrak{h}) \subset H
\]
**Definition:** A vector field \( X \) is **strongly tangent** to a subgroup \( H \), if there exists a \( k \in \mathbb{N} \) and a smooth \( k \)-parameter family \( \{ \phi(t_1,\ldots,t_k) \} \) of diffeomorphisms in \( H \) such that

1. \( \phi(t_1,\ldots,t_k) = \text{Id}, \) if \( t_j = 0 \) for some \( 1 \leq j \leq k; \)

2. \[ \frac{\partial^k \phi(t_1,\ldots,t_k)}{\partial t_1 \ldots \partial t_k} \bigg|_{(t_1,\ldots,t_k)=(0,\ldots,0)} = X. \]

**Proposition:** The Lie algebra generated by strongly tangent vector fields is tangent to \( H \).

\[
X_1 \text{ strongly tangent } \Rightarrow \{ \phi^1_{(t_1,\ldots,t_{k_1})} \} \quad \Rightarrow \quad [\phi^1_{(t_1,\ldots,t_{k_1})}, \phi^2_{(t_1,\ldots,t_{k_2})}]
\]

\[
X_2 \text{ strongly tangent } \Rightarrow \{ \phi^2_{(t_1,\ldots,t_{k_2})} \}
\]

\[
[\phi^1_{(t_1,\ldots,t_{k_1})}, \phi^2_{(t_1,\ldots,t_{k_2})}] = \left( \phi^1_{(t_1,\ldots,t_{k_1})} \right)^{-1} \circ \left( \phi^2_{(t_1,\ldots,t_{k_2})} \right)^{-1} \circ \left( \phi^1_{(t_1,\ldots,t_{k_1})} \right) \circ \left( \phi^2_{(t_1,\ldots,t_{k_2})} \right)
\]

\[
[\phi^1_{(t_1,\ldots,t_{k_1})}, \phi^2_{(t_1,\ldots,t_{k_2})}] \quad \Rightarrow \quad [X_1, X_2]
\]
Tangent Lie algebras to the $\text{Hol}_x(M)$

**Proposition:** $\mathcal{K}_x(M)$ and $\mathfrak{hol}^*_x(M)$ are tangent to $\text{Hol}_x(M)$.

- $\mathcal{K}(M)$: the *curvature algebra* is the smallest Lie algebra generated by curvature vector fields.

- $\mathfrak{hol}^*(M)$: the *infinitesimal holonomy algebra* is the smallest Lie algebra generated by curvature vector fields and by horizontal Berwald differentiation.
Projectively flat Finsler surfaces of constant curvature

Remarks:  
• \( \dim M = 2 \),  
• \( \mathcal{I}_x \simeq S^1 \),  
• \( \text{Hol}_x(M) \subset \text{Diff}_+^{\infty}(S^1) \)  
• \( \dim \mathcal{R}_x(M) \leq 1 \),  
• \( \text{hol}^*_x(M) \) can be higher (even infinite) dimensional,  
• \( G^i = \mathcal{P}(x, y)y^i \),  
• \( R^i_{jk} = \lambda \left( \delta^i_j g_{km}y^m - \delta^i_k g_{jm}y^m \right) \).
**Theorem.** The holonomy group of a projectively flat, spherically symmetric Finsler 2-manifolds of constant curvature is maximal:

$$ \text{Hol}(\bar{M}) = \text{Diff}^\infty_+(S^1). $$

**Proof:** there exists $x_o \in M$, where $\mathcal{F}(x_o, y) = \|y\|$ and $\mathcal{P}(x_o, y) = c \cdot \|y\|

- $\mathcal{I}_{x_o} = S^1$, $\text{Hol}_o(M) \subset \text{Diff}^\infty_+(S^1)$
- $\text{hol}_{x_o}^*(M) \supset \mathcal{F}(S^1) = \left\{ \cos nt \frac{\partial}{\partial t}, \sin nt \frac{\partial}{\partial t} \right\}_{n \in \mathbb{N}} \Rightarrow \mathcal{F}(S^1) = \text{hol}_{x_o}^*(M) = \mathcal{X}(S^1)$
- $\exp(\mathcal{X}(S^1)) = \exp(\text{hol}_{x_o}^*(M)) \subset \exp(\text{hol}_{x_o}^*(M)) \subset \text{Diff}^\infty_+(S^1)$
- $\left\langle \exp(\mathcal{X}(S^1)) \right\rangle \subset \left\langle \exp(\text{hol}_{x_o}^*(M)) \right\rangle \subset \text{Hol}_o(M) \subset \text{Diff}^\infty_+(S^1)$
- $\left\langle \exp(\mathcal{X}(S^1)) \right\rangle$ conj. inv. $\Rightarrow$ normal subgroup in $\text{Diff}^\infty_+(S^1)$
- $\text{Diff}^\infty_+(S^1)$ simple

$$ \Rightarrow \left\langle \exp(\mathcal{X}(S^1)) \right\rangle = \text{Diff}^\infty_+(S^1) \Rightarrow \text{Hol}_o(M) = \text{Diff}^\infty_+(S^1) $$

**Corollary:** The holonomy group of the Funk metric (constant negative curvature) and of the Bryant-Shen 2-spheres (constant positive curvature) are maximal.
Projectively flat Finsler surfaces of constant curvature

**Theorem:** The holonomy group of a locally projectively flat Finsler surface of constant curvature is finite dimensional if and only if

1. $R = 0$, 

2. $R \neq 0$ and the associated canonical connection is linear.

- If $\lambda \neq 0$, $\nabla$ is nonlinear: suppose that $\text{Hol}(M)$ is finite dimensional:

  S. Lie: If a finite-dimensional connected Lie group acts on a 1-dimensional manifold without fixed points, than its dimension is less than 4.

  - $x_0 \in M$, $\xi = R_{x_0}(X, Y)$
  - $\nabla$ nonlinear $\Rightarrow \{\xi, \nabla_1 \xi, \nabla_2 \xi\}$ $\mathbb{R}$-linearly independent,
  - $\{\xi, \nabla_1 \xi, \nabla_2 \xi, \nabla_i \nabla_j \xi\}$ $\mathbb{R}$-linearly dependent,
  - $\left\{1, \frac{\partial P}{\partial y^1}, \frac{\partial P}{\partial y^2}, 2\frac{\partial P}{\partial y^i} \frac{\partial P}{\partial y^j} - \lambda g_{ij}\right\}$ $\mathbb{R}$-linearly dependent
  - $\lambda g_{ij} = 2P_i P_j + A_{ij} + B_{ij}^m P_m$, $A_{ij}, B_{ij} \in \mathbb{R}$,
  - $g_{ij} = \partial_{y^i y^j} E \Rightarrow \partial_i g_{jk} - \partial_k g_{ij} = 0 \Rightarrow$ PDE on $P$
\[\begin{align*}
2\mathcal{P}_2\mathcal{P}_{11} - 2\mathcal{P}_1\mathcal{P}_{12} + b_2\mathcal{P}_{11} + (c_2 - b_1)\mathcal{P}_{12} - c_1\mathcal{P}_{22} &= 0, \\
2\mathcal{P}_1\mathcal{P}_{22} - 2\mathcal{P}_2\mathcal{P}_{12} - b_3\mathcal{P}_{11} + (b_2 - c_3)\mathcal{P}_{12} + c_2\mathcal{P}_{22} &= 0.
\end{align*}\]

\[\mathcal{P}(x_0, y) = y_2 \cdot f\left(\frac{y_1}{y_2}\right) \Rightarrow \begin{cases}
f''\left(\frac{2}{y_2} f + \frac{b_2}{y_2} + (b_1 - c_2) \frac{y_1}{y_2} + \frac{c_1 y_1^2}{y_2^3}\right) = 0, \\
f''\left(\frac{2y_1}{y_2} f - \frac{b_3}{y_2} + (c_3 - b_2) \frac{y_1}{y_2} + \frac{c_2 y_1^2}{y_2^3}\right) = 0.
\end{cases}\]

\[t = \frac{y_1}{y_2} \Rightarrow \begin{cases}
2f + b_2 + (b_1 - c_2)t + c_1 t^2 = 0, \\
2tf - b_3 + (c_3 - b_2)t + c_2 t^2 = 0.
\end{cases}\]

\[b_3 + (2b_2 - c_3)t - (2c_2 - b_1)t^2 + c_1 t^3 \equiv 0,\]

\[f(t) = -b_2 - c_2t, \quad \mathcal{P}(x_0, y) = -y_2b_2 - c_2y_1\]

\[\Rightarrow \mathcal{P} \text{ is linear} \Rightarrow \nabla \text{ is linear} \Rightarrow \text{contradiction.}\]

\[\Rightarrow \text{Hol}_{x_0}(M) \text{ is infinite dimensional.}\]
References


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