Holonomy distribution and degree of metrizability

of a SODE

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Calculus of variations:

- scalar product $\langle v, v \rangle = E(v)$  
  Riemann: $\langle v, v \rangle_x$  
  Finsler: $\langle v, v \rangle_{(x,y)}$

- $I[\gamma] = \int_\gamma E(\gamma, \dot{\gamma})$

\[
\frac{d}{dt} \frac{\partial E}{\partial \dot{x}^i} - \frac{\partial E}{\partial x^i} = 0, \quad \Rightarrow \quad \dot{x}^i \frac{\partial^2 E}{\partial x^i \partial \dot{x}^j} + \ddot{x}^i \frac{\partial^2 E}{\partial \dot{x}^i \partial \dot{x}^j} - \frac{\partial E}{\partial x^j} = 0 \quad \Rightarrow \quad \ddot{x}^i = f^i(x, \dot{x}),
\]

Metric $\Rightarrow$ SODE

The inverse problem: metrizability

Metric $\Leftarrow? SODE$

Remark: metrizability $\iff$ Paolo: dynamical potential?
Geometric tools associated to a SODE:

- SODE $\implies$ Spray: vectorfield on $TM$ $S = (x^i, y^i, \dot{y}^i, f^i(x, y))$

- Path of the spray: $\gamma$ $S_{\dot{\gamma}} = \ddot{\gamma} \iff \frac{d^2 x^i}{dt^2} = f^i \left( x, \frac{dx}{dt} \right)$.

- Parallel translation...

The Euler-Lagrange PDE system

\[
\ddot{x}^i = f^i(x, \dot{x}) \implies \begin{cases}
\omega_E = y^i \frac{\partial^2 E}{\partial x^i \partial y^j} + f^i \frac{\partial^2 E}{\partial y^i \partial y^j} - \frac{\partial E}{\partial x^j} = 0, \\
y^i \frac{\partial E}{\partial y^i} - 2E = 0,
\end{cases}
\]

Euler-Lagrange functions

\[\mathcal{E}_S = \{ E \mid \omega_E = 0 \}, \quad \mathcal{E}_{S,2} = \{ E \mid \omega_E = 0, \mathcal{L}_C E = 2E \},\]
Parallel translation: geometric construction

\[ \tau \circ \tau \rightarrow \tau \circ \tau \]

\[ \tau : v \rightarrow w \]

\[ w = \tau (v) \]
- $R \equiv 0$

- $R \not\equiv 0$
Holonomy distribution

\[ \mathcal{H} := \langle HTM \rangle_{\text{Lie}} \]

\[ \mathcal{H} = HTM \oplus v\mathcal{H}, \quad \text{Im } R \subset v\mathcal{H}, \]

\[ C^\infty_{\text{hol}} = \{ E \mid \mathcal{L}_X E = 0, \ X \in \mathcal{H} \}, \quad C^\infty_{\text{hol}, 2} = \{ E \in C^\infty_{\text{hol}} \mid \mathcal{L}_C E = 2E \} \]

**Proposition:** \( \mathcal{E}_{S, 2} = C^\infty_{\text{hol}, 2} \)

**Property:** \( \mathcal{E}_{S, C^\infty_{\text{hol}}}, \mathcal{E}_{S, 2}(= C^\infty_{\text{hol}, 2}) \) are vector space over \( \mathbb{R} \).

**Proposition:** A 1-homogeneous functional combination of 2-homogeneous Euler-Lagrange functions is a 2-homogeneous Euler-Lagrange function.

\[ E(x, y) := \varphi(E_1(x, y), \ldots, E_r(x, y)) \]

\[ \mathcal{L}_X E = \frac{\partial \varphi}{\partial z^1} \cdot \mathcal{L}_X E_1 + \cdots + \frac{\partial \varphi}{\partial z^r} \cdot \mathcal{L}_X E_r = 0, \quad \forall X \in \mathcal{H} \]
Definition: The *degree of metric freedom* $m_S$ of a metrizable spray $S$ is the maximal number of functionally independent elements of $E_{S,2}$. If the spray $S$ is non-metrizable, then we set $m_S = 0$.

**Theorem:** If $S$ is metrizable and $\mathcal{H}$ is regular, then

$$m_S = \text{codim} \mathcal{H}.$$

**Remark:** From the hypothesis of the Theorem one cannot omit the metrizability. There are examples for not metrizable sprays with $\text{codim} \mathcal{H} > 0$.

**Corollary:** If $S$ is isotropic, then

- $m_S = 0$ if and only if $R \neq 0$ and $S$ is not metrizable;
- $m_S = 1$ if and only if $R \neq 0$ and $S$ is metrizable;
- $m_S = n$ if and only if $R = 0$. 
Explicite examples

- codim $\mathcal{H} = 0$, $m_S = 0$: $f^i := \sqrt{x^2(y^1)^2 + (y^2)^2 y^i + (-1)^i y^1 y^i_{2x^2}}$,

- codim $\mathcal{H} = 1$, $m_S = 1$: $f^i = \frac{\mu \langle x, y \rangle}{1 + \mu |x|^2} y^i$, $\mu \in \mathbb{R} \setminus \{0\}$,

- codim $\mathcal{H} = n$, $m_S = n$: $f^i = \frac{\langle a, y \rangle}{1 + \langle a, x \rangle} y^i$,

- codim $\mathcal{H} > 0$, $m_S = 0$: $f^1 = -\frac{(y^1)^2}{2x^2}$, $f^2 = 0$. 
References


Thank You