Classification of Semisimple Lie Algebras

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General overview

ODE → Lie group ↔ Lie algebra

Simple groups (of Lie type) ↔ (Semi)simple Lie algebras

Simple groups

Finite State Machines
Introduction to Lie algebras

- Classical Lie algebras are algebras with Lie bracket or commutator: \([x,y] = xy - yx\)

**Example** \((gl(2, \mathbb{C}), sl(2, \mathbb{C}))\). Let the general linear algebra \(gl(2, \mathbb{C})\) over the complex numbers \(\mathbb{C}\) consist of the \(2 \times 2\) matrices:

\[
gl(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C} \right\}.
\]

**Special linear algebra** \(sl(2, \mathbb{C})\)

\[
sl(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.
\]
Abstract Lie algebras

Definition  A vector space \( L \) over a field \( F \) with a bilinear operation \( L \times L \to L \) denoted by \( (x, y) \mapsto [x, y] \) (called Lie bracket) is called a Lie algebra over \( F \) if the following axioms are satisfied:

1. \([x, y] = -[y, x]\) for all \( x, y \in L \),

2. \([x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\) for all \( x, y, z \in L \). This is called the Jacobian identity.

A subset \( S \) of a Lie algebra \( L \) is called a Lie subalgebra if \( S \) is a subspace which is closed under the Lie bracket.

Let \( S_1, \ldots, S_n \) be subspaces or subalgebras of a Lie algebra \( L \). Then \( L \) is the direct sum of \( S_i \), that is

\[ L = \bigoplus_{i=1}^n S_i, \]

if and only if every element of \( L \) can be uniquely written as \( \sum_{i=1}^n s_i \), where \( s_i \in S_i \).
Abstract Lie algebras

A homomorphism $\phi: L_1 \to L_2$ is a linear map satisfying $\phi([x, y]) = [\phi(x), \phi(y)]$.

$\text{im } \phi = \{ \varphi(x) \mid x \in L_1 \} \subseteq L_2$

$\text{ker } \phi = \{ x \in L_1 \mid \phi(x) = 0 \} \subseteq L_1$

The quotient Lie algebra $L/I$ is the factor space (as quotient vector space) with elements $\{ x + I \mid x \in L \}$, and let us define the Lie bracket as $[x + I, y + I] = [x, y] + I$.

Theorem (Homomorphism theorem). Let $L_1, L_2$ be Lie algebras over the field $F$ and let $\phi: L_1 \to L_2$ be a homomorphism. Then

$L_1/\ker \phi \simeq \text{im } \phi.$
Abstract Lie algebras

The derived subalgebra \([L, L]\) is the smallest Lie subalgebra of \(L\) containing all elements \([x, y]\) (for all \(x, y \in L\)).

The center of a Lie algebra \(L\) (denoted by \(Z(L)\)) consists of those elements with which the Lie bracket is identically zero:

\[
Z(L) = \{ z \in L : [z, x] = 0 \text{ for all } x \in L \}. 
\]

A Lie algebra \(L\) called Abelian if every Lie-bracket is zero, that is \(L = Z(L)\). The center of an arbitrary Lie algebra is Abelian.

If a Lie algebra \(L\) is not Abelian (\([L, L] \neq 0\)) and \(L\) has no nontrivial ideals then we call \(L\) is simple.

For any element \(x \in L\) of a Lie algebra \(L\) let \(\text{ad}_x : L \to L\) be the linear map defined by \(\text{ad}_x(y) = [x, y]\). s the adjoint representation \(\text{ad} : L \to \mathfrak{gl}(L)\), \(\text{ad} : x \mapsto \text{ad}_x\).
Abstract Lie algebras

Define a sequence of ideals of $L$ called the derived series by $L^0 = L, L^1 = [L, L], L^2 = [L^1, L^1], \ldots, L^i = [L^{i-1}, L^{i-1}]$. Let us call $L$ solvable if $L^n = 0$ for some $n$.

Every simple Lie algebra is nonsolvable, since $L^n = L$ for every positive integer $n$.

If $L$ does not contain any non-zero solvable ideal then $L$ is called semisimple.

An $L$ is semisimple if and only if $L$ is a direct sum of simple Lie-algebras.
Cartan decomposition

If $L$ is semisimple, then we have root space decomposition or Cartan decomposition, that is $L$ is the direct sum (as subspaces) of $H$ and of the subspaces $L_r$:  
\[ L = H \oplus (\bigoplus_{r \in \Phi} L_r). \]

$H$ is an Abelian subalgebra called Cartan subalgebra  
\{ $\text{ad}_h \mid h \in H$ \} is simultaneously diagonalizable  
$L_r = \{ x \in L \mid [h, x] = r(h)x \text{ for all } h \in H \}$ where $r: H \to \mathbb{C}$ is a linear map  
$H^*$ is the dual space of $H$ that is the vectorspace of all linear functions from $H$ to $\mathbb{C}$  
The set of all nonzero (linear functions) $r \in H^*$ for which $L_r \neq 0$ is denoted by $\Phi$. The elements of $\Phi$ are called roots and $\Phi$ is called a root system.
Example \((sl(2, \mathbb{C}))\). Let \(L = sl(2, \mathbb{C})\). Then

\[
H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} : \alpha \in \mathbb{C} \right\}
\]

is a Cartan subalgebra. Note that every Cartan subalgebra can be written as \(P^{-1}HP\) where \(P\) is an invertible matrix of size \(2 \times 2\).

Let \(h = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \in H\) and \(x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\). Now,

\[
[h, x] = \begin{pmatrix} 0 & 2\alpha b \\ -2\alpha c & 0 \end{pmatrix}.
\]

That is, \(ad_h\) (for \(h \in H\)) is diagonal in the basis \(e_{11} - e_{22}, e_{12}, e_{21}\):

\[
ad_h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}.
\]
That is, $\text{ad}_h$ (for $h \in H$) is diagonal in the basis $e_{11} - e_{22}, e_{12}, e_{21}$:

$$\text{ad}_h = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2\alpha & 0 \\
0 & 0 & -2\alpha
\end{pmatrix}.$$ 

Consequently, the Cartan decomposition of $L$ is

$$L = H \oplus L_r \oplus L_{-r},$$

where

$$L_r = \{ \lambda e_{12} \mid \lambda \in \mathbb{C} \},$$

$$L_{-r} = \{ \lambda e_{21} \mid \lambda \in \mathbb{C} \},$$

and

$$r : H \rightarrow \mathbb{C} \quad r(h) = r \left( \begin{pmatrix}
\alpha & 0 \\
0 & -\alpha
\end{pmatrix} \right) = 2\alpha.$$

Now, $\dim H = 1$, $\dim L_r = 1$, $\dim L_{-r} = 1$ and the root system is $\Phi = \{ r, -r \}$. 

BIOMICS
Killing form

If $x, y \in L$, define $\kappa(x, y) = \text{Tr}(\text{ad}_x \text{ad}_y)$. Then $\kappa$ is a symmetric bilinear form on $L$, called the **Killing form**.

The Killing form $\kappa$ is associative in the sense that $\kappa([x, y], z) = \kappa(x, [y, z])$.

Let $L$ be a Lie-algebra. Then $L$ is semisimple if and only if its Killing form is nondegenerate.

Since the restriction of the Killing form $\kappa$ to $H$ is nondegenerate we can identify $H$ with $H^*$: to every $r \in H^*$ corresponds a unique element $t_r \in H$ satisfying

$$r(h) = \kappa(t_r, h) \text{ for all } h \in H.$$

Now, we can introduce a scalar product $(.,.)$ on $H^*$ such that

$$(r, s) = \kappa(t_r, t_s) \text{ for all } r, s \in H^*.$$
Abstract root systems

Let $V$ be a real Euclidean space of finite dimension $l$ with scalar product $(\cdot, \cdot)$. Let $w_r$ denote the reflection to the hyperplane orthogonal to the non-zero vector $r$. That is, for $x \in V$ we have

$$w_r(x) = x - \frac{2(r,x)}{(r,r)} r.$$ 

Now, $w_r(r) = -r$ and $w_r(y) = y$ for all $y$ with $(r,y) = 0$.

1. $\Phi$ is a set of non-zero vectors.
2. $\Phi$ spans $V$.
3. If $r, s \in \Phi$ then $w_r(s) \in \Phi$.
4. If $r, s \in \Phi$ then $2(r,s)/(r,r)$ is an integer.
5. If $r, \lambda r \in \Phi$, where $\lambda \in \mathbb{R}$ then $\lambda = \pm 1$.

The rank of a root system $\Phi$ is the dimension of $V$ (spanned by $\Phi$): \[ \text{rank } \Phi = \dim V = l. \]

$$A_{sr} = \frac{2(r,s)}{(s,s)} = 2\frac{||r||}{||s||} \cos \theta.$$
Root systems of rank 2

Classifying root systems of rank 2 we obtain only four different root systems. $A_1 \times A_1$ is a reducible root system, $A_2$, $B_2$ and $G_2$ are irreducible root systems.
A root system $\Phi$ is called **indecomposable** (or **irreducible**) if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to every root in the other set by the scalar product $(\cdot, \cdot)$.

A semisimple $L$ Lie algebra is simple if and only if the corresponding root system $\Phi$ is indecomposable.

**Existence theorem** Let $\Phi$ be an indecomposable root system. Then there exists a simple Lie algebra over $\mathbb{C}$ which has a root system equivalent to $\Phi$.

**Isomorphism theorem.** Any two simple Lie algebras over $\mathbb{C}$ with equivalent root systems are isomorphic.
Let $\Phi$ be a system of roots. A subset $\Pi \subset \Phi$ is called a fundamental root system (or base) if the following axioms are satisfied.

1. $\Pi$ is linearly independent.

2. Every root in $\Phi$ is a linear combination of roots in $\Pi$ with coefficients which are either all non-negative or all non-positive.

If $\Pi$ is a fundamental system in $\Phi$ then $(r, s) \leq 0$ for all different elements $r, s$ in $\Pi$. 
Dynkin diagrams

- Infinite series

\[ A_l \ (l \geq 1) : \quad 1 \quad 2 \quad 3 \quad \cdots \quad l-1 \quad l \]

\[ B_l \ (l \geq 2) : \quad 1 \quad 2 \quad \cdots \quad l-2 \quad l-1 \quad l \]

\[ C_l \ (l \geq 3) : \quad 1 \quad 2 \quad \cdots \quad l-2 \quad l-1 \quad l \]

\[ D_l \ (l \geq 4) : \quad 1 \quad 2 \quad \cdots \quad l-3 \quad l-2 \quad \cdots \]

\[ \vdots \]
Dynkin diagrams

\[ E_6 : \]
\[
\begin{array}{cccccc}
1 & 2 & 4 & 5 & 6 & 3 \\
\end{array}
\]

\[ E_7 : \]
\[
\begin{array}{ccccccccc}
1 & 2 & 4 & 5 & 6 & 7 & 3 \\
\end{array}
\]

\[ E_8 : \]
\[
\begin{array}{cccccccc}
1 & 2 & 4 & 5 & 6 & 7 & 8 & 3 \\
\end{array}
\]

\[ F_4 : \]
\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

\[ G_2 : \]
\[
\begin{array}{cccc}
1 & 2 \\
\end{array}
\]